Math 2413  Homework Set 6 – Solutions  10 Points

1. (2 pts)

\[ 2xe^{y^2} + 2yy'x^2e^{y^2} = 8 - 18yy' \]
\[ \left( 2yx^2e^{y^2} + 18y \right)y' = 8 - 2xe^{y^2} \quad \Rightarrow \quad y' = \frac{8 - 2xe^{y^2}}{2yx^2e^{y^2} + 18y} \]

4. (2 pts) We’ll need both: \( V = \frac{4}{3} \pi r^3 \) & \( S = 4\pi r^2 \). We know \( V' = -3 \) and we want to determine \( r' \) when \( S = 35 \). So, first use the surface area to determine the radius at the time in question.

\[ 35 = 4\pi r^2 \quad \Rightarrow \quad r^2 = \frac{35\pi}{4} \quad \text{or} \quad r = 1.6689 \text{cm} \]

Now, from the volume formula we get,
\[ V' = 4\pi r^2 r' \quad \Rightarrow \quad -3 = 35 r' \quad \Rightarrow \quad r' = -\frac{3}{35} = -0.08571 \]

Notice that, in this case, we didn’t actually need the radius as the formula for the surface area just happened to show up in the derivative. Note that this will usually NOT happen!

5. (2 pts) A sketch of the tank with some water in it is to the right. We know that \( V' = 2 \) and we want to find \( h' \) when \( h = 20 \).

The volume of the tank is \( V = \frac{1}{3} \pi r^2 h \) and we can use similar triangles to eliminate the \( r \) from the equation so we can get everything in terms of \( h \) (which we’ll need to do or we will end up with an \( r' \) in the derivative which we know nothing about).

Similar triangles gives us,
\[ \frac{r}{h} = \frac{12}{35} \quad \Rightarrow \quad r = \frac{12}{35} h \quad \Rightarrow \quad V = \frac{48}{1225} \pi h^3 \]

We can now proceed as normal.
\[ V' = \frac{144}{1225} \pi h^2 h' \quad \Rightarrow \quad 2 = \frac{144}{1225} \pi (20)^2 h' \quad \Rightarrow \quad h' = \frac{49}{1122} \pi = 0.01354 \text{ cm/s} \]

10. (2 pts)

\[ A'(t) = \frac{-\sin(t)}{1 + \cos(t)} \quad A''(t) = \frac{-\cos(t)(1 + \cos(t)) - \sin^2(t)}{(1 + \cos(t))^2} = \frac{-\cos(t) + 1}{(1 + \cos(t))^2} = \frac{-1}{1 + \cos(t)} \]

11. (2 pts)

\[ g'(x) = \frac{1}{6} \cos\left(\frac{x}{6}\right) + \frac{80e^{19}}{4x^3} + 3e^{-3x} = \frac{1}{6} \cos\left(\frac{x}{6}\right) + \frac{20}{x^3} + 3e^{-3x} \]
\[ g''(x) = -\frac{1}{36} \sin\left(\frac{x}{6}\right) - \frac{20}{x^2} - 9e^{-3x} \quad g^{(3)}(x) = -\frac{1}{216} \cos\left(\frac{x}{6}\right) + \frac{40}{x^3} + 27e^{-3x} \]
2. \[
\frac{3x^2 - 2yy'}{x^3 - y^2} = 3x^2y' + 2x^2y''\]
\[
\left(2x^2y' + \frac{2yy'}{x^3 - y^2}\right)y' = \frac{3x^2}{x^3 - y^2} - 3x^2y^2 \quad \Rightarrow \quad y' = \frac{3x^2}{x^3 - y^2} - \frac{3x^2y^2}{2x^2yy' + \frac{2yy'}{x^3 - y^2}}
\]

3. To answer this question we'll need to do some implicit differentiation to get the derivative, which we will in turn need for the slope of the tangent line.
\[
(y^3 + 3xy^2y')e^{xy^3} = 4y^3y'
\]
\[
(4y^3 - 3xy^2e^{xy^3})y' = y^3e^{xy^3} \quad y' = \frac{y^3e^{xy^3}}{4y^3 - 3xy^2e^{xy^3}} = \frac{ye^{xy^3}}{4y - 3xe^{xy^3}}
\]
The tangent line is then,
\[
m = y'|_{x=0, y=-2} = \frac{1}{4} \quad y = -2 + \frac{1}{4}x
\]

6. Here is the sketch for each part of this problem and notice that for (c) we've actually moved past the starting point of boat B.

In each case we're going to need to find \( z' \) and to do this we'll use the Pythagorean Theorem and so we may as well set that up now and then we'll actually work the problem.
\[
x^2 + y^2 = z^2 \quad \Rightarrow \quad z' = \frac{1}{z}(xx' + yy')
\]
Note as well that all times will need to be converted to seconds since the speeds are all in seconds...
(a) Here’s all the important quantities for this part.
\[ x = 1000 - 40(10) = 600 \quad x' = -40 \quad y = 25(10) = 250 \quad y' = 25 \]
\[ z = \sqrt{600^2 + 250^2} = 650 \]
The rate at which the distance between the two boats is changing is,
\[ z' = \frac{1}{650} \left((600)(-40) + (250)(25)\right) = -27.3077 \text{ m/hr} \]
So, in this case the distance is **decreasing**.

(b) Here’s all the important quantities for this part.
\[ x = 1000 - 40(20) = 200 \quad x' = -40 \quad y = 25(20) = 500 \quad y' = 25 \]
\[ z = \sqrt{200^2 + 500^2} = 538.5165 \]
The rate at which the distance between the two boats is changing is,
\[ z' = \frac{1}{538.5165} \left((200)(-40) + (500)(25)\right) = 8.3563 \text{ m/hr} \]
So, in this case the distance is **increasing**.

(c) Here’s all the important quantities for this part.
\[ x = 40(30) - 1000 = 200 \quad x' = 40 \quad y = 25(30) = 750 \quad y' = 25 \]
\[ z = \sqrt{200^2 + 750^2} = 776.2087 \]
The rate at which the distance between the two boats is changing is,
\[ z' = \frac{1}{776.2087} \left((200)(40) + (750)(25)\right) = 34.4624 \text{ m/hr} \]
So, in this case the distance is **increasing**.

7. With this problem we’ve been given the equation already. We’ll need to determine the value of \( P \) at the time in question so let’s do that first.
\[ P^2 - 4P = 7(1-1)e^{6-7t} = 0 \quad \rightarrow \quad P(P-4) = 0 \quad \rightarrow \quad P \neq 0 \quad \text{or} \quad P = 4 \]
Note that we can exclude the first possibility because we were told that, at the time in question, \( P \) is positive. Now, we also know that \( z' = -2 \) and so all we need to do is some implicit differentiation and then plug in known quantities.
\[ 2PP' - 4P' = 7ze^{6-7z} - 7(z-1)e^{6-7z}(7z') \]
\[ (2(4)-4)P' = 7(-2)e^{-1} \quad \Rightarrow \quad P' = -\frac{2}{7} e^{-1} = -1.2876 \]
So, \( P \) must be decreasing since \( P' < 0 \).

8. \( f(x) = 8x^4 + \frac{4}{x} x^2 + 81x^{\frac{1}{3}} \quad f'(x) = 32x^3 - \frac{4}{x^3} x^3 + 27x^{-\frac{2}{3}} \quad f''(x) = 96x^2 + 4x^{-4} - 18x^{-\frac{5}{3}} \)
9. \[ \frac{dy}{dz} = -3z^2 \sin(z^3) + 5 \csc^2(5z) \]

\[ \frac{d^2y}{dz^2} = -6z \sin(z^3) - 9z^4 \cos(z^3) - 50 \csc^2(5z) \cot(5z) \]