3. (2 pts) A quick substitution \((u = \sin x)\) will put this integral into a form that is more easily recognizable.

\[
\int \frac{(1 + \sin x) \cos x}{\sin^2 x - 14 \sin x + 24} \, dx = \int \frac{1+u}{u^2 - 14u + 24} \, du = \int \frac{1+u}{(u-2)(u-12)} \, du
\]

This is now clearly an integral that can be done with partial fractions. I’ll leave the details of the partial fractions to you since it is a simple partial fractions. Here is the integral.

\[
\int \frac{(1 + \sin x) \cos x}{\sin^2 x - 14 \sin x + 24} \, dx = \int \frac{\frac{13}{10}}{u-12} \, du - \int \frac{\frac{3}{10}}{u-2} \, du = \frac{13}{10} \ln |\sin (x) - 12| - \frac{3}{10} \ln |\sin (x) - 2| + c
\]

5. (2 pts) You had one almost exactly like this in the integration by parts homework set. The only difference is that it involved a cosine instead of an exponential. It works the same way however.

\[
u = t^4 \quad dv = t^3 e^{t^2} dt \quad \Rightarrow \quad du = 4t^3 dt \quad v = \frac{1}{4} e^{t^2}
\]

The integral is then,

\[
\int t^7 e^{t^2} \, dt = \frac{1}{4} t^4 e^{t^2} - \int t^3 e^{t^2} \, dt = \frac{1}{4} t^4 e^{t^2} - \frac{1}{2} e^{t^2} + c
\]

7. (2 pts) Note that this integral has a discontinuity at \(y = 3\) so we’ll need to break up the integral at that point. Also, the integral is a simple partial fractions integral so I’ll let you check my work.

\[
\int_1^4 \frac{1}{y^2 - 3y} \, dy = \int_1^3 \frac{1}{y^2 - 3y} \, dy + \int_3^4 \frac{1}{y^2 - 3y} \, dy
\]

\[
\begin{align*}
= \lim_{t \to 3} \int_1^t \frac{1}{y^2 - 3y} \, dy + \lim_{t \to 3} \int_s^4 \frac{1}{y^2 - 3y} \, dy \\
= \lim_{t \to 3} \left( \frac{1}{2} \ln |y - 3| - \frac{1}{2} \ln |y| \right)_1^t + \lim_{t \to 3} \left( \frac{1}{2} \ln |y - 3| - \frac{1}{2} \ln |y| \right)_s^4 \\
= \lim_{t \to 3} \left( \frac{1}{2} \ln |t - 3| - \ln |t| - \ln (2) \right) + \lim_{t \to 3} \left( \frac{1}{2} \ln (4) - \ln |s - 3| + \ln |s| \right) \\
= \left[ -\infty - \frac{1}{2} \ln (3) - \frac{1}{2} \ln (2) \right] + \left[ -\frac{1}{2} \ln (4) + \infty + \frac{1}{3} (3) \right]
\end{align*}
\]

So, both of the individual integrals are divergent (although we only need one to be divergent...) and so the whole integral is also divergent. Note that the two infinities do NOT cancel!

8. (2 pts) This integral is unlike anything we did in class. We have both an infinite limit as well as a discontinuity. We can deal with this by breaking up the limit at any point (I’ll use \(z = 1\)) and then we’ll have two integrals that we can do using the substitution \(u = -\frac{1}{2}z\).

\[
\begin{align*}
\int_0^\infty \frac{e^{-\frac{1}{2}z}}{z^2} \, dz &= \int_0^1 \frac{e^{-\frac{1}{2}z}}{z^2} \, dz + \int_1^\infty \frac{e^{-\frac{1}{2}z}}{z^2} \, dz \\
&= \lim_{t \to 0} \int_0^t \frac{e^{-\frac{1}{2}z}}{z^2} \, dz + \lim_{s \to \infty} \int_s^1 \frac{e^{-\frac{1}{2}z}}{z^2} \, dz \\
&= \lim_{t \to 0} \left( e^t - e^{-\frac{1}{2}} \right) + \lim_{s \to \infty} \left( e^{-\frac{1}{2}} - e^1 \right) = (e^t - 0) + (1 - e^1) = 1
\end{align*}
\]
So, both of the individual integrals were convergent and so the integral we were asked to do is also convergent and its value is 1.

10. (2 pts) In this case the two trig functions won’t change things much and so based on the $x$ I’m going to guess that this diverges and so I’ll need a smaller function that diverges.

$$\frac{x^2 + \cos^2 x}{x^3 \sin^4 x} \geq \frac{x^2 + 0}{x^3 \sin^4 x} \quad \text{b/c } \cos^2 x \geq 0$$

$$\geq \frac{x^2}{x^3 (1)} \quad \text{b/c } \sin^4 x \leq 1$$

$$= \frac{1}{x}$$

Finally, we know that \( \int_{\frac{5}{3}}^{\infty} \frac{1}{x} \, dx \) diverges and so by the Comparison Test the original integral must also diverge.

---

1. \(\int \cos^3 (1-2\theta) \sin^6 (1-2\theta) \, d\theta = \int (1-\sin^2 (1-2\theta)) \sin^6 (1-2\theta) \cos (1-2\theta) \, d\theta\)

$$= -\frac{1}{2} \int (1-u^2) u^6 \, du = \frac{1}{8} \sin^8 (1-2\theta) - \frac{1}{12} \sin^7 (1-2\theta) + c$$

2. In order to do this integral we’ll first need to square things out.

$$\int (t + 2 \cos (3t))^2 \, dt = \int t^2 + 4t \cos (3t) + 4 \cos^2 (3t) \, dt$$

Now split this into three separate integrals. The first is easy, the second is an integration by parts and the third can be done with a half angle formula. All three of these are fairly basic integrals so I’ll leave the details of each to you to check. Here is what you should get.

$$\int (t + 2 \cos (3t))^2 \, dt = \frac{1}{3} t^3 + \frac{6}{3} t \sin (3t) + \frac{2}{3} \cos (3t) + 2 \left( t - \frac{1}{2} \sin (6t) \right) + c$$

4. This is a trig substitution problem, but also needs a regular substitution \((u = e^{2z})\) to put it into that form. As I did in the trig substitution section I’ll combine the two into one substitution.

$$e^{2z} = \sin \theta \quad 2e^{2z} \, dz = \cos \theta \, d\theta \quad \sqrt{1 - e^{4z}} = \sqrt{1 - (e^{2z})^2} = \sqrt{1 - \sin^2 \theta} = |\cos \theta| = \cos \theta$$

Using this the integral becomes,

$$\int e^{2z} \sqrt{1 - e^{4z}} \, dz = \frac{1}{2} \int \cos^2 \theta \, d\theta = \frac{1}{4} \int 1 + \cos (2\theta) \, d\theta = \frac{1}{4} \theta + \frac{1}{8} \sin (2\theta) + c$$
Now we need to go back to \( z \)'s. However, the right triangle we get from the trig substitution will only allow us to convert trig functions involving arguments of \( \theta \) only and we have \( 2\theta \). A double angle formula will fix this up. The right triangle tells us that \( \cos \theta = \sqrt{1 - e^{-4z}} \) and so the integral is,

\[
\int e^{2z} \sqrt{1 - e^{-4z}} \, dz = \frac{1}{2} \theta + \frac{1}{2} \cos \theta \sin \theta + c = \frac{1}{2} \sin^{-1} \left( e^{2z} \right) + \frac{1}{2} e^{2z} \sqrt{1 - e^{-4z}} + c
\]

6. The integral here is a simple integration by parts that I’m leaving the details to you to check.

\[
\int_{-\infty}^{0} 10xe^{2x} \, dx = \lim_{t \to -\infty} \int_{t}^{0} 10xe^{2x} \, dx = \lim_{t \to -\infty} \left( \frac{55}{2} - \frac{5}{2} (2t - 1) e^{2t} \right) = \frac{55}{2} e^{12}
\]

The limit of the second term can be done with L’Hospitals Rule (you do remember that right?). Since the limit exists and is finite we know that the integral is \textit{convergent} and has a value of \( \frac{55}{2} e^{12} \).

9. In this case the \( x \) in the denominator seems to suggest that this is divergent. However, the \( e^{-x} \) in the numerator is going to zero so fast that it will override everything and so I’m going to guess that this will converge. So, we’ll need something larger that we know converges. There are several ways to do this, but a nice way of dealing with things is to note that because of our interval of integration we know that \( x \geq 1 \) and so if we replace the \( x \) in the denominator with 1 (which we know is always smaller than or equal to \( x \) the denominator will get smaller and hence the fraction will get larger. Or,

\[
\frac{e^{-x}}{x+2} < \frac{e^{-x}}{1+2} = \frac{1}{3} e^{-x}
\]

Now, we can easily show (which you need to do, but since I did this in class will skip the details here) that \( \int_{1}^{\infty} \frac{1}{3} e^{-x} \, dx \) converges and so by the Comparison Test the original integral will also converge.

11. For this problem we have \( f(x) = \ln(x^4 + 1) \) and \( \Delta x = \frac{1}{2} \).

**MidPoint Rule**

\[
\int_{1}^{3} \ln(x^4 + 2) \, dx \approx \frac{1}{2} \left[ f(1.25) + f(1.75) + f(2.25) + f(2.25) \right] = 5.4607
\]

**Trapezoid Rule**

\[
\int_{1}^{3} \ln(x^4 + 2) \, dx \approx \frac{1}{4} \left[ f(1) + 2 f(1.5) + 2 f(2) + f(2.5) + f(3) \right] = 5.4379
\]

**Simpson’s Rule**

\[
\int_{1}^{3} \ln(x^4 + 2) \, dx \approx \frac{1}{6} \left[ f(1) + 4 f(1.5) + 2 f(2) + 4 f(2.5) + f(3) \right] = 5.4561
\]
For comparison’s sake,

\[ \int_{1}^{3} \ln(x^4 + 2) \, dx = 5.452878081 \]