2. (3 pts) First determine y limits. Using the fact that we’re assuming that \( y \geq 0 \) we get \( 0 \leq y \leq 2 \).

\[
\frac{dx}{dy} = 2y \quad \quad L = \int_{0}^{2} \sqrt{1 + 4y^2} \, dy
\]

This integral will need the trig substitution \( y = \frac{1}{2} \tan \theta \). I’ll leave the most of the details to you to verify.

\[
L = \int_{0}^{2} \sqrt{1 + 4y^2} \, dy = \frac{1}{2} \tan^{-1}(4) \int_{0}^{\pi/2} \sec^3 \theta \, d\theta = \frac{1}{2} \left( \sec \theta \tan \theta + \ln|\sec \theta + \tan \theta| \right)_{0}^{\pi/2} = 4.6468
\]

4. (2 pts)

\[
\frac{dy}{dx} = -2x \quad \quad ds = \sqrt{1 + 4x^2} \, dx
\]

\[
A = \int 2\pi x \, ds = \int_{0}^{2} 2\pi x \sqrt{1 + 4x^2} \, dx = \frac{\pi}{6} \left( 17^2 - 1 \right) = 36.1769
\]

8. (2 pts)

\[
y = \frac{2}{e^{-x^2}} = \frac{2}{x^2}
\]

Because we know that exponentials are always positive we know that \( x > 0 \) and \( y > 0 \). Now, \( e^{-x} \) is decreasing (negative exponent) and \( t > 0 \) we also know that \( x < 1 \). Next \( e^{2t} \) is increasing (positive exponent) and again \( t > 0 \) we also know that \( y > 2 \). All together this is,

\[
0 < x < 1 \quad \quad 2 < y < \infty
\]

This means we will only have the right portion of the graph. As \( t \) increases we see that \( x \) decreases and \( y \) increases and the curve will only be traced out once.

11. (3 pts) \( y = 3 \left( \cos^2(4t) \right)^3 = 3x^3 \) From the equations we see that,

\[
0 \leq x \leq 1 \quad \quad 0 \leq y \leq 3
\]

so we’ll get a portion of this curve that repeats itself because both \( x \) and \( y \) will oscillate.

Next, \( t \)’s for the endpoints.

\[
x = 0 : \quad \cos^2(4t) = 0 \rightarrow \cos(4t) = 0 \rightarrow 4t = \frac{\pi}{2} + 2\pi n \quad or \quad 4t = \frac{3\pi}{2} + 2\pi n
\]

\[
\rightarrow \quad t = \frac{\pi}{8} + \frac{1}{2} \pi n \quad or \quad t = \frac{3\pi}{8} + \frac{1}{2} \pi n
\]

\[
x = 1 : \quad \cos^2(4t) = 1 \rightarrow \cos(4t) = \pm 1 \rightarrow 4t = 0 + 2\pi n \quad or \quad 4t = \pi + 2\pi n
\]

\[
\rightarrow \quad t = \frac{\pi}{4} n \quad or \quad t = \frac{3\pi}{4} + \frac{1}{2} \pi n
\]

Plugging in some value of \( n \) we will have the following range for one trace: \( 0 \leq t \leq \frac{\pi}{8} \).

This means that the curve will be traced out \( \frac{11\pi - 0}{\frac{\pi}{8} - 0} = 88 \) times.
1. \[
\frac{dy}{dx} = -\frac{1}{2} (x - 2)^{\frac{1}{3}} \quad \quad L = \int_2^{10} \sqrt{1 + \frac{9}{4} (x - 2)} \, dx = \int_2^{10} \frac{3}{2} \sqrt{\frac{9}{4} x - \frac{7}{2}} \, dx = \frac{1}{27} (152 \sqrt{19} - 8) = 24.2427
\]

3. This is not as difficult as it looks. This is just an ellipse and so we know that the limits on \(x\) and \(y\) are,
\[-2 \leq x \leq 2 \quad \quad -5 \leq y \leq 5\]
So, we’ll have limits once we get the integral set up. To do that we only need to solve for \(x\) or \(y\) and in this case it doesn’t really matter which we solve for. I’ll do solve for \(y\) since that is what most are used to solving for.
\[
y = \pm \sqrt{25 \left(1 - \frac{1}{4} x^2\right)} = \pm 5 \sqrt{1 - \frac{1}{4} x^2}
\]
The “+” will give the upper portion of the ellipse and the “-” will give the lower. Also since we can only use a single equation to find the arc length and because ellipses are symmetric we can find the length of the top and then just double that to get the complete length. This will also mean that we’ll use \(x\) limits of integration. Here’s the rest of the work.
\[
\frac{dy}{dx} = \frac{5}{2} \left(1 - \frac{1}{4} x^2\right)^{\frac{1}{2}} \left(-\frac{1}{2} x\right) = -\frac{5}{4} \frac{x}{\sqrt{1 - \frac{1}{4} x^2}}
\]
\[
\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{25}{16} \frac{x^2}{1 - \frac{1}{4} x^2}} = \frac{5}{4} \sqrt{\frac{16 + 21x^2}{25 \left(1 - \frac{1}{4} x^2\right)}} \quad \quad \Rightarrow \quad \quad L = 2 \int_{-2}^{2} \frac{5}{4} \sqrt{\frac{16 + 21x^2}{25 \left(1 - \frac{1}{4} x^2\right)}} \, ds
\]

5. \[
\frac{dx}{dy} = \frac{6 - 2y}{2\sqrt{6y - y^2}} \quad \quad ds = \sqrt{1 + \left(\frac{6-2y}{6y-y^2}\right)^2} \, dy = \sqrt{\frac{36}{4(6y-y^2)}} \, dy = \frac{6}{2\sqrt{6y-y^2}} \, dy = \frac{3}{\sqrt{6y-y^2}} \, dy
\]
\[
A = \int 2\pi x \, ds = \int_{1}^{4} 2\pi \sqrt{6y-y^2} \frac{3}{\sqrt{6y-y^2}} \, dy = \int_{1}^{4} 6\pi \, dy = 18\pi
\]

6. Not much to this one, but both will use the same \(ds\) to make things easier.
\[
\frac{dy}{dx} = 3 \sec^2 (3x) \quad \quad ds = \sqrt{1 + 9 \sec^4 (3x)} \, dx
\]
(a) \[
A = \int 2\pi y \, ds = \int_{0}^{\frac{\pi}{2}} 2\pi \tan (3x) \sqrt{1 + 9 \sec^4 (3x)} \, dx
\]
(b) \[
A = \int 2\pi x \, ds = \int_{0}^{\frac{\pi}{2}} 2\pi x \sqrt{1 + 9 \sec^4 (3x)} \, dx
\]
7. \[ y = (t+2)^2 - 3(t+2) = t^2 + t - 2 \] So, we have a parabola with vertex \((-\frac{1}{2}, -\frac{9}{4})\). The limits for \(x\) and \(y\) are then, \[ -\infty < x < \infty \quad -\frac{9}{4} \leq y < \infty \] As \(t\) increases we see that \(x\) also increases and so the direction is left to right and because \(x\) will only increase as \(t\) increases we can see that the curve will be traced out once.

9. \[ \cos^2(3t) = y \quad \sin^2(3t) = \frac{1}{2}(x^2 - 1) \] \[ 1 = \sin^2(3t) + \cos^2(3t) = y + \frac{1}{2}x^2 - \frac{1}{2} \quad \Rightarrow \quad y = \frac{3}{2} - \frac{1}{2}x^2 \] From the parametric equations we have, \[ 0 \leq \sin^2(3t) \leq 1 \rightarrow 0 \leq 2\sin^2(3t) \leq 2 \rightarrow 1 \leq 1 + 2\sin^2(3t) \leq 3 \] \[ \rightarrow 1 \leq \sqrt{1 + 2\sin^2(3t)} \leq \sqrt{3} \rightarrow 1 \leq x \leq \sqrt{3} \] \[ 0 \leq \cos^2(3t) \leq 1 \rightarrow 0 \leq y \leq 1 \] This means we will only have a portion of the parabola as shown to the right. Also, this curve will retrace itself as \(t\) increases since both \(x\) and \(y\) will oscillate as \(t\) increases.

Finally, \(t\)’s for the endpoints.
\[ y = 0 : \cos^2(3t) = 0 \rightarrow \cos(3t) = 0 \rightarrow 3t = \frac{\pi}{2} + 2\pi n \quad \text{or} \quad 3t = \frac{3\pi}{2} + 2\pi n \] \[ \rightarrow \frac{t}{3} = \frac{\pi}{6} + \frac{\pi}{2} n \quad \text{or} \quad t = \frac{\pi}{2} + \frac{2}{3} n \] \[ y = 1 : \cos^2(3t) = 1 \rightarrow \cos(3t) = \pm 1 \rightarrow 3t = \pi + 2\pi n \quad \text{or} \quad 3t = \pi + 2\pi n \] \[ \rightarrow t = \frac{\pi}{3} n \quad \text{or} \quad t = \frac{\pi}{3} + \frac{2}{3} \pi n \] Plugging in some value of \(n\) we will have the following range for one trace: \[ 0 \leq t \leq \frac{\pi}{6} \]

10. \[ \cos^2 \left( \frac{\pi}{4} \right) = y^2 \quad \sin^2 \left( \frac{\pi}{4} \right) = \frac{\pi^2}{16} \] \[ 1 = \sin^2 \left( \frac{\pi}{4} \right) + \cos^2 \left( \frac{\pi}{4} \right) = \frac{\pi^2}{16} + y^2 \] Next, notice that the curve will “start” (i.e. \(t = 0\)) at the point \((0,1)\). Now, some derivatives of the equations are,
\[
\frac{dx}{dt} = \frac{5}{4} \cos\left(\frac{t}{4}\right) \quad \quad \quad \frac{dy}{dt} = -\frac{1}{4} \sin\left(\frac{t}{4}\right)
\]

So, if we start at \( t = 0 \) and increase \( t \) we can see that initially, \( \frac{dx}{dt} > 0 \) and so \( x \) must increase while \( \frac{dy}{dt} < 0 \) and so \( y \) must decrease (which pretty much had to happen given where we start. The \( x \) increasing however, says that we must be moving in the clockwise direction as shown in the sketch.

The limits on \( x \) and \( y \) are easy to see from the parametric equations and are,

\[-5 \leq x \leq 5 \quad \quad -1 \leq y \leq 1\]

Finally, \( t \)'s for the endpoints.
\[
x = 5 : \; \sin\left(\frac{t}{4}\right) = 1 \; \rightarrow \; \frac{t}{4} = \frac{\pi}{2} + 2\pi n \; \rightarrow \; t = 2\pi + 8\pi n
\]

Plugging in some value of \( n \) we will have the following range for one trace:

\[
-6\pi \leq t \leq 2\pi \quad \text{or} \quad 2\pi \leq t \leq 10\pi
\]

This means that the curve will be traced out \( \frac{12\pi - 0}{10\pi - 2\pi} = \frac{3}{2} = 1.5 \) times.