#2. (2 pts) First the complimentary solution is (I’ll leave the details to you to check).
\[ y_c(t) = c_1 + c_2 e^{-2t} \]

The guess for the particular solution (and its derivatives) is,
\[ Y_p(t) = Ae^{-t} + t(Bt + C) = Ae^{-t} + Bt^2 + Ct \]
\[ Y'_p(t) = -Ae^{-t} + 2Bt + C \]
\[ Y''_p(t) = Ae^{-t} + 2B \]

Note that we needed to add an extra \( t \) to the second and third terms since a constant is part of the complimentary solution. Plugging into the differential equation and simplifying gives,
\[ -4Ae^{-t} + 14Bt + 6B - 7C = 9 - e^{-t} - 14t \]

Setting coefficient equal and solving gives,
\[ e^{-t} : \quad -4A = -1 \quad \Rightarrow \quad A = \frac{1}{4} \]
\[ t^1 : \quad 14B = -14 \quad \Rightarrow \quad B = -1 \]
\[ t^0 : \quad 6B - 7C = 9 \quad \Rightarrow \quad C = \frac{5}{7} \]

The general solution is then,
\[ y(t) = c_1 + c_2 e^{2t} + \frac{1}{4} e^{-t} - t^2 + \frac{5}{7} t \]

#4. (2 pts) First the complimentary solution is (I’ll leave the details to you to check).
\[ y_c(t) = c_1 e^{6t} \cos t + c_2 e^{6t} \sin t \]

The guess for the particular solution is,
\[ Y_p(t) = (At^4 + Bt^3 + Ct^2 + Dt + E) e^{6t} + t(F e^{6t} \cos t + Ge^{6t} \sin t) \]

The second term needs an extra \( t \) because it is exactly the complimentary solution without the \( t \).

#7. (2 pts) First the complimentary solution is (I’ll leave the details to you to check).
\[ y_c(t) = c_1 e^{\frac{1}{2}t} + c_2 e^{-3t} \]

Next, \( g(t) \) and the Wronskian.
\[ g(t) = 2 + \frac{3}{2} e^{2t} \]
\[ W = \begin{vmatrix} e^{\frac{1}{2}t} & e^{-4t} \\ \frac{1}{2} e^{\frac{1}{2}t} & -4e^{-4t} \end{vmatrix} = -\frac{9}{2} e^{-2t} \]

The particular solution is then,
\[ Y_p = -e^{\frac{1}{2}t} \int \frac{e^{-4t} \left( 2 + \frac{3}{2} e^{2t} \right)}{-\frac{9}{2} e^{-2t}} dt + e^{-4t} \int \frac{e^{\frac{1}{2}t} \left( 2 + \frac{3}{2} e^{2t} \right)}{-\frac{9}{2} e^{-2t}} dt \]
\[ = \frac{2}{9} e^{\frac{1}{2}t} \left( -4e^{-2t} + e^{\frac{3}{2}t} \right) - \frac{7}{9} e^{-4t} \left( \frac{1}{2} e^{4t} + \frac{3}{2} e^{6t} \right) = -1 + \frac{1}{6} e^{2t} \]

The general solution is then,
\[ y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^{-3t} - 1 + \frac{1}{6} e^{2t} \]
Finally apply the initial conditions, solve for the constants and find the actual solution.

\[
c_1 + c_2 - 1 + \frac{1}{6} = -2 \quad c_1 = -\frac{8}{9} \quad c_2 = -\frac{5}{18} \quad \Rightarrow \quad y(t) = -\frac{8}{9}e^{-\frac{t}{18}}e^{-4t} - 1 + \frac{1}{6}e^{2t}
\]

**#9. (2 pts)** Here are the important quantities.

\[
m = 0.06 \quad k = \frac{9.8(0.06)}{0.01} = 58.8 \quad \gamma = \frac{0.75}{0.0625} = 12 \quad \gamma_{CR} = 2\sqrt{(58.8)(0.06)} = 3.7566
\]

It looks like we’re overdamped in this case. The IVP is,

\[
0.06u'' + 12u' + 58.8u = 0 \quad u(0) = 0.10 \quad u'(0) = -0.08
\]

The general solution to this is,

\[
u(t) = c_1e^{-100-2\sqrt{2255}t} + c_2e^{100-2\sqrt{2255}t} = c_1e^{-5.0263t} + c_2e^{-194.9737t}
\]

Applying the initial conditions gives the actual solution of,

\[
u(t) = 0.1022e^{-5.0263t} - 0.002225e^{-194.9737t}
\]

**#11. (2 pts)** The IVP for this case is (using the previous work from #9).

\[
0.06u'' + 12u' + 58.8u = 7\cos(2t) \quad u(0) = 0.10 \quad u'(0) = -0.08
\]

The complimentary solution from #9 is,

\[
u_c(t) = c_1e^{-5.0263t} + c_2e^{-194.9737t}
\]

The form of the particular solution will be,

\[
U_p(t) = A\cos(2t) + B\sin(2t)
\]

Differentiating this, plugging into the differential equation and simplifying will give,

\[
(58.56A + 24B)\cos(2t) + (-24A + 58.56B)\sin(2t) = 7\cos(2t)
\]

Setting coefficients equal gives,

\[
\begin{align*}
\cos(2t) & : 58.56A + 24B = 7 \\
\sin(2t) & : -24A + 58.56B = 0
\end{align*}
\]

\[
\Rightarrow A = 0.10235 \quad B = 0.04194
\]

So, the particular and general solutions are then,

\[
U_p(t) = 0.10235\cos(2t) + 0.04194\sin(2t)
\]

\[
u(t) = c_1e^{-5.0263t} + c_2e^{-194.9737t} + 0.10235\cos(2t) + 0.04194\sin(2t)
\]

Applying the initial conditions gives,

\[
u(t) = -0.0032699e^{-5.0263t} + 0.00092487e^{-194.9737t} + 0.10235\cos(2t) + 0.04194\sin(2t)
\]

The final step is to combine the last two terms into a single cosine.

\[
R = \sqrt{(0.10235)^2 + (0.04194)^2} = 0.11061 \quad \delta_1 = \tan^{-1}\left(\frac{0.04194}{0.10235}\right) = 0.389001 \quad \delta_1 = 0.389001
\]

\[
\delta_2 = \delta_1 + \pi = 3.53059
\]

In this case we need \(\delta_1\) and so the final answer is,

\[
u(t) = -0.0032699e^{-5.0263t} + 0.00092487e^{-194.9737t} + 0.11061\cos(2t - 0.389001)\]
#1. First the complimentary solution is (I’ll leave the details to you to check).

\[ y_c(t) = c_1 e^{-4t} \cos(2t) + c_2 e^{-4t} \sin(2t) \]

The guess for the particular solution (and its derivatives) is,

\[ Y_p(t) = A + B \cos(2t) + D \sin(2t) \]
\[ Y'_p(t) = -2B \sin(2t) + 2D \cos(2t) \]
\[ Y''_p(t) = -4B \cos(2t) - 4D \sin(2t) \]

Note that we don’t need an extra \( t \) here because the sine and cosine both have an exponential in front of them in the complimentary solution. Plugging into the differential equation and simplifying gives,

\[ 20A + (16B + 16D) \cos(2t) + (-16B + 16D) \sin(2t) = 5 + 4 \cos(2t) - 8 \sin(2t) \]

Setting coefficient equal and solving gives,

\[ \cos(2t) : \quad 16B + 16D = 4 \quad A = \frac{1}{4} \]
\[ \sin(2t) : \quad -16B + 16D = -8 \quad \Rightarrow \quad B = \frac{1}{8} \]
\[ t^0 : \quad 10A = 5 \quad D = -\frac{1}{8} \]

The general solution is then,

\[ y(t) = c_1 e^{-4t} \cos(2t) + c_2 e^{-4t} \sin(2t) + \frac{1}{4} + \frac{1}{8} \cos(2t) - \frac{1}{8} \sin(2t) \]

#3. First the complimentary solution is (I’ll leave the details to you to check).

\[ y_c(t) = c_1 e^{-4t} + c_2 e^{2t} \]

The guess for the particular solution (and its derivatives) is,

\[ Y_p(t) = Ae^t + t(Bl + C)e^{2t} = Ae^t + (Bl^2 + Ct)e^{2t} \]
\[ Y'_p(t) = Ae^t + (2Bl^2 + (2B + 2C)t + C)e^{2t} \]
\[ Y''_p(t) = Ae^t + (4Bl^2 + (8B + 4C)t + 2B + 4C)e^{2t} \]

Note that we had to add an extra \( t \) onto the second term because \( Ce^{2t} \) is part of the complimentary solution. Plugging into the differential equation and simplifying gives,

\[ 12Bte^{2t} + (2B + 6C)e^{2t} - 5Ae^{-4t} = 20e^t - 90e^{2t} \]

Setting coefficient equal and solving gives,

\[ te^{2t} : \quad 12B = -90 \quad A = -4 \]
\[ e^{2t} : \quad 2B + 6C = 0 \quad \Rightarrow \quad B = -\frac{15}{2} \]
\[ e^t : \quad -5A = 20 \quad C = \frac{5}{2} \]

The general solution is then,

\[ y(t) = c_1 e^{-4t} + c_2 e^{2t} - 4e^t + \frac{5}{2}(t - 3t^2)e^{2t} \]

Finally, apply the initial conditions, solve for the constants and get the actual solution.
\[ c_1 + c_2 - 4 = 0 \quad c_1 = \frac{17}{12} \]
\[ -4c_1 + 2c_2 - 4 + \frac{5}{2} = -2 \quad c_2 = \frac{11}{12} \]
\[ y(t) = \frac{17}{12} e^{-4t} + \frac{31}{12} e^{2t} - 4e^t + \frac{5}{2} \left( t - 3t^2 \right) e^{2t} \]

\#5. First the complimentary solution is (I’ll leave the details to you to check).
\[ y_c(t) = c_1 e^{-12t} + c_2 t e^{-12t} \]
The guess for the particular solution is,
\[ Y_p = (A t + B) \cos (4t) + (C t + D) \sin (4t) + t^2 \left( E t^2 + F t + G \right) e^{-12t} \]
The second term needs an extra \( t^2 \) because with no \( t \) or a single \( t \) a term from the complimentary solution is buried in it.

\#6. First the complimentary solution is (I’ll leave the details to you to check).
\[ y_c(t) = c_1 e^{-\frac{3}{2}t} \cos (2t) + c_2 e^{-\frac{3}{2}t} \sin (2t) \]
Next, \( g(t) = 3e^{-\frac{3}{2}t} \) (recall, the \( y'' \) needs a coefficient of 1) and the Wronskian is,
\[ W = \begin{vmatrix} e^{-\frac{3}{2}t} \cos (2t) & e^{-\frac{3}{2}t} \sin (2t) \\ -\frac{3}{2} e^{-\frac{3}{2}t} \cos (2t) - 2e^{-\frac{3}{2}t} \sin (2t) & -\frac{3}{2} e^{-\frac{3}{2}t} \sin (2t) + 2e^{-\frac{3}{2}t} \cos (2t) \end{vmatrix} \]
\[ = -\frac{3}{2} e^{-t} \sin (2t) \cos (2t) + 2e^{-t} \cos^2 (2t) - \left( -\frac{3}{2} e^{-t} \cos (2t) \sin (2t) - 2e^{-t} \sin^2 (2t) \right) \]
\[ = 2e^{-t} \cos^2 (2t) + 2e^{-t} \sin^2 (2t) = 2e^{-t} \]
The particular solution is then,
\[ Y_p = -e^{-\frac{3}{2}t} \cos (2t) \int e^{-\frac{3}{2}t} \sin (2t) \left( 3e^{-\frac{3}{2}t} \right) \frac{dt}{2e^{-t}} + e^{-\frac{3}{2}t} \sin (2t) \int e^{-\frac{3}{2}t} \cos (2t) \left( 3e^{-\frac{3}{2}t} \right) \frac{dt}{2e^{-t}} \]
\[ = -\frac{3}{4} e^{-\frac{3}{2}t} \cos (2t) \int \sin (2t) dt + \frac{3}{4} e^{-\frac{3}{2}t} \sin (2t) \int \cos (2t) dt \]
\[ = \frac{3}{4} e^{-\frac{3}{2}t} \cos^2 (2t) + \frac{3}{4} e^{-\frac{3}{2}t} \sin^2 (2t) = \frac{3}{4} e^{-\frac{3}{2}t} \]
The general solution is then,
\[ y(t) = c_1 e^{-\frac{3}{2}t} \cos (2t) + c_2 e^{-\frac{3}{2}t} \sin (2t) + \frac{3}{4} e^{-\frac{3}{2}t} \]

\#8. Here are all the important quantities.
\[ m = \frac{1}{32} = \frac{1}{128} \quad k = \frac{1}{6} = \frac{1}{2} \quad \omega_0 = \frac{1}{\sqrt{1/128}} = 8 \]
The IVP is,
\[ \frac{1}{128} u'' + \frac{1}{6} u = 0 \quad u(0) = -\frac{1}{12} \quad u'(0) = -\frac{5}{12} \]
The general solution is
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\[ u(t) = c_1 \cos(8t) + c_2 \sin(8t) \]

Applying the initial conditions gives,
\[ u(t) = -\frac{1}{12} \cos(8t) - \frac{5}{96} \sin(8t) \]

Now convert into a single cosine.
\[ R = \sqrt{\left(\frac{1}{12}\right)^2 + \left(\frac{5}{96}\right)^2} = \frac{\sqrt{65}}{96} \]
\[ \delta_1 = \tan^{-1} \left(\frac{-\frac{5}{96}}{-\frac{1}{12}}\right) = 0.5586 \]
\[ \delta_2 = \delta_1 + \pi = 3.7002 \]

In this case \( \delta \) is in the third quadrant and so \( \delta_2 \) is the correct angle. The solution is,
\[ u(t) = \frac{\sqrt{65}}{96} \cos(8t - 3.7002) \]

**#10.** Here’s the IVP for this case (using the previous work from #8).
\[ \frac{1}{12} u'' + \frac{1}{2} u = e^{-4t} - \sin t \]
\[ u(0) = -\frac{1}{12}, \quad u'(0) = -\frac{5}{12} \]

From #8 we know that the complimentary solution is,
\[ u_c(t) = c_1 \cos(8t) + c_2 \sin(8t) \]

We’ll use undetermined coefficients for the particular solution. The form will be,
\[ U_p(t) = A \cos t + B \sin t + Ce^{-4t} \]

Note that at this point we know that we WON’T have resonance. Because the frequency in the forcing function sine is not \( \omega = 8 \) we won’t need to add a \( t \) onto these terms and so we won’t get resonance. Differentiating \( U_p(t) \), plugging into the differential equation and simplifying gives,
\[ \cos t : \frac{63}{128} A \cos(8t) + \frac{63}{128} B \sin(8t) + \frac{5}{8} Ce^{-4t} = e^{-4t} - \sin t \]

Setting coefficients equal gives,
\[
\begin{align*}
\cos t &: \quad \frac{63}{128} A = 0 \\
\sin t &: \quad \frac{63}{128} B = -1 \\
e^{-4t} &: \quad \frac{5}{8} C = 1
\end{align*}
\]

\[ \Rightarrow \quad A = 0 \quad B = -\frac{128}{63} \quad C = \frac{8}{5} \]

The particular and general solution is then,
\[ U_p(t) = -\frac{128}{63} \sin t + \frac{8}{5} e^{-4t} \]
\[ u(t) = c_1 \cos(8t) + c_2 \sin(8t) - \frac{128}{63} \sin t + \frac{8}{5} e^{-4t} \]

Applying the initial conditions gives,
\[ u(t) = -1.68333 \cos(8t) + 1.00188 \sin(8t) - \frac{128}{63} \sin t + \frac{8}{5} e^{-4t} \]

The final step is to then combine the first two terms into a single cosine.
\[ R = \sqrt{(-1.68333)^2 + (1.00188)^2} = 1.9589 \]
\[ \delta_1 = \tan^{-1} \left(\frac{1.00188}{-1.68333}\right) = -0.5369 \]
\[ \delta_2 = \delta_1 + \pi = 2.6047 \]
We’ll need $\delta_2$ here so the final answer is,

$$u(t) = 1.9589 \cos(8t - 2.6047) - \frac{128}{63} \sin t + \frac{8}{5} e^{-4t}$$