2. (2 pts)

\[ W = \begin{vmatrix} x^2 & g \\ \frac{1}{3} x^\frac{3}{2} & g' \end{vmatrix} = x^\frac{7}{2} g' - \frac{1}{3} x^\frac{3}{2} g = x^3 e^{2x} \rightarrow g' - \frac{1}{27} g = x^\frac{7}{2} e^{2x} \]

This is a simple linear differential equation so I’ll let you verify that the solution is,

\[ g(x) = \frac{1}{2} x^\frac{7}{2} e^{2x} + cx^\frac{5}{2} \]

4. (3 pts) I’ll let you verify the complimentary solution: \( y_c(t) = c_1 e^{ct} \cos(3t) + c_2 e^{ct} \sin(3t) \)

The guess for the particular solution and its derivatives are,

\[ Y_p = At^3 + Bt^2 + Ct + D \quad Y'_p = 3At^2 + 2Bt + C \quad Y''_p = 6At + 2B \]

Plugging this into the differential equation and simplifying gives,

\[ 10At^3 + (6A + 10B)t^2 + (6A + 4B + 10C)t + 2B + 2C + 10D = 6t - 20t^3 \]

Setting coefficients equal gives,

\[ t^3: \quad 10A = -20 \quad A = -2 \]
\[ t^2: \quad 6A + 10B = 0 \quad B = \frac{3}{5} \]
\[ t: \quad 6A + 4B + 10C = 6 \quad C = \frac{33}{25} \]
\[ t^0: \quad 2B + 2C + 10D = 0 \quad D = -\frac{61}{125} \]

The general solution is then,

\[ y(t) = c_1 e^{ct} \cos(3t) + c_2 e^{ct} \sin(3t) - 2t^3 + \frac{6}{5} t^2 + \frac{33}{25} t - \frac{61}{125} \]

5. (2 pts) I’ll let you verify that the complimentary solution is: \( y_c(t) = c_1 e^{-6t} + c_2 e^{4t} \).

The guess for the particular solution and its derivatives are,

\[ Y_p = Ae^{-3t} \quad Y'_p = -3Ae^{-3t} \quad Y''_p = 9Ae^{-3t} \]

Plugging this into the differential equation and simplifying gives,

\[ -21Ae^{-3t} = -8e^{-3t} \quad \Rightarrow \quad -21A = -8 \quad A = \frac{8}{21} \]

The general solution is then,

\[ y(t) = c_1 e^{-6t} + c_2 e^{4t} + \frac{8}{21} e^{-3t} \]

7. (3 pts)

I’ll leave it to you to verify that the complimentary solution is: \( y_c(t) = c_1 \cos(3t) + c_2 \sin(3t) \)

The guess for the particular solution and its derivatives are,
\[ Y_p = (At + B)\cos(2t) + (Dt + E)\sin(2t) \]
\[ Y_p' = (A + 2E + 2Dt)\cos(2t) + (D - 2B - 2At)\sin(2t) \]
\[ Y_p'' = (4D - 4B - 4At)\cos(2t) + (-4A - 4E - 4Dt)\sin(2t) \]

Plugging this into the differential equation and simplifying gives,
\[ (5B + 4D + 5At)\cos(2t) + (-4A + 5E + 5Dt)\sin(2t) = 8t\sin(2t) \]

Setting coefficients equal gives,
\[
\begin{align*}
t \cos(2t): & \quad 5A = 0 \\
cos(2t): & \quad 5B + 4D = 0 \\
t \sin(2t): & \quad 5D = 8 \\
sin(2t): & \quad -4A + 5E = 0
\end{align*}
\]

The general solution is then,
\[ y(t) = c_1 \cos(3t) + c_2 \sin(3t) - \frac{32}{25} \cos(2t) + \frac{8}{5} t \sin(2t) \]

Applying the initial conditions gives,
\[
\begin{align*}
c_1 - \frac{32}{25} &= 2 \\
3c_2 &= -5
\end{align*}
\]

The actual solution is then,
\[ y(t) = \frac{32}{25} \cos(3t) - \frac{5}{3} \sin(3t) - \frac{32}{25} \cos(2t) + \frac{8}{5} t \sin(2t) \]

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**Not Graded**

1. All we need to do is show that the Wronskian is not zero.

\[
W = \begin{vmatrix}
e^{\lambda t} \cos(\mu t) & e^{\lambda t} \sin(\mu t) \\
\lambda e^{\lambda t} \cos(\mu t) - \mu e^{\lambda t} \sin(\mu t) & \lambda e^{\lambda t} \sin(\mu t) + \mu e^{\lambda t} \cos(\mu t)
\end{vmatrix}
\]

\[
= e^{\lambda t} \cos(\mu t)(\lambda e^{\lambda t} \sin(\mu t) + \mu e^{\lambda t} \cos(\mu t)) - e^{\lambda t} \sin(\mu t)(\lambda e^{\lambda t} \cos(\mu t) - \mu e^{\lambda t} \sin(\mu t))
\]

\[
= \mu e^{2\lambda t} \cos^2(\mu t) + \mu e^{2\lambda t} \sin^2(\mu t) = \mu e^{2\lambda t} \neq 0 \quad \text{provided} \ \mu \neq 0
\]

The Wronskian is not zero (provided \(\mu \neq 0\), but that must be true in order to have complex roots) and so \(y_1(t) = e^{\lambda t} \cos(\mu t)\) and \(y_2(t) = e^{\lambda t} \sin(\mu t)\) are a fundamental set of solutions and a general solution in this case is \(y(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)\) as I claimed.

3. I'll leave it to you to verify that the complimentary solution is:

\[ y_c(t) = c_1 e^{-\frac{1}{2}t} + c_2 t e^{-\frac{1}{2}t} \]

The guess for the particular solution and its derivatives are,
\[ Y_p = A \cos\left(\frac{t}{2}\right) + B \sin\left(\frac{t}{2}\right) \quad Y'_p = -\frac{1}{4} A \sin\left(\frac{t}{2}\right) + \frac{1}{2} B \cos\left(\frac{t}{2}\right) \]
\[ Y''_p = -\frac{1}{4} A \cos\left(\frac{t}{2}\right) - \frac{1}{4} B \sin\left(\frac{t}{2}\right) \]

Plugging this into the differential equation and simplifying gives,
\[ \left( -\frac{21}{4} A + 5B \right) \cos\left(\frac{t}{2}\right) + \left( -5A - \frac{21}{4} B \right) \sin\left(\frac{t}{2}\right) = 58 \cos\left(\frac{t}{2}\right) \]

Setting coefficients equal gives,
\[
\begin{align*}
\cos\left(\frac{t}{2}\right) : & \quad -\frac{21}{4} A + 5B = 58 \\
\sin\left(\frac{t}{2}\right) : & \quad -5A - \frac{21}{4} B = 0
\end{align*}
\]

\[ A = -\frac{168}{29} \quad B = \frac{160}{29} \]

The general solution is then,
\[
y(t) = c_1 e^{-t} + c_2 t e^{-t} - \frac{168}{29} \cos\left(\frac{t}{2}\right) + \frac{160}{29} \sin\left(\frac{t}{2}\right)\]

6. I’ll leave it to you to verify that the complimentary solution is : \[ y_c(t) = c_1 e^{-5t} + c_2 e^{-2t} \]

The guess for the particular solution and its derivatives are,
\[ Y_p = \left( A t^2 + B t + C \right) e^t \quad Y'_p = \left( A t^2 + (2A + B) t + B + C \right) e^t \]
\[ Y''_p = \left( A t^2 + (4A + b) t + 2A + 2B + C \right) e^t \]

Plugging this into the differential equation and simplifying gives,
\[ \left(2A + 9B + 18C\right) e^t + \left(18A + 18B\right) t e^t + 18At^2 e^{-t} = 15t^2 e^t \]

Setting coefficients equal gives,
\[
\begin{align*}
t^2 e^{-t} : \quad & \quad 18A = 15 \quad \Rightarrow \quad A = \frac{5}{6} \\
t e^{-t} : \quad & \quad 18A + 18B = 0 \quad \Rightarrow \quad B = -\frac{5}{6} \\
e^{-t} : \quad & \quad 2A + 9B + 18C = 0 \quad \Rightarrow \quad C = \frac{35}{108}
\end{align*}
\]

The general solution is then,
\[
y(t) = c_1 e^{-5t} + c_2 e^{-2t} + \left(\frac{5}{6} t^2 - \frac{5}{6} t + \frac{35}{108}\right) e^{-t}\]