Math 2413 **Homework Set 11 – Solutions** **10 Points**

1. (2 pts) $u = 8-2t^2 \qquad du = -4t\,dt \qquad t\,dt = -\frac{1}{4}du \qquad\qquad t=0 : u=8, t=2 : u=0$

$$\int_0^2 2t\,\mathbf{e}^{8-2t^2}\,dx = -\frac{1}{2}\int_8^0 \mathbf{e}^u\,du = -\frac{1}{2}\mathbf{e}^u\Big|_8^0 = \boxed{-\frac{1}{2}+\frac{1}{2}\mathbf{e}^8}$$

3. (2 pts) In this case we'll need to split the integral up since on the second term needs a substitution.

$$\int_1^4 12z+\sin\left(\tfrac{\pi z}{2}\right)dz = \int_1^4 12z\,dz + \int_1^4 \sin\left(\tfrac{\pi z}{2}\right)dz$$

$$u = \tfrac{\pi z}{2} \qquad du = \tfrac{\pi}{2}dz \qquad dz = \tfrac{2}{\pi}du \qquad\qquad z=1 : u=\tfrac{\pi}{2}, \;\; u=4 : u=2\pi$$

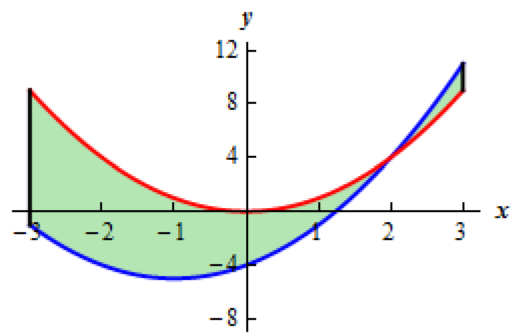
$$\int_1^4 12z+\sin\left(\tfrac{\pi z}{2}\right)dz = \int_1^4 12z\,dz + \tfrac{2}{\pi}\int_{\frac{\pi}{2}}^{2\pi}\sin(u)\,du = 6z^2\Big|_1^4 - \tfrac{2}{\pi}\cos(u)\Big|_{\frac{\pi}{2}}^{2\pi} = \boxed{90-\tfrac{2}{\pi}}$$

7. (2 pts) First we'll need intersection points.

$$x^2+2x-4 = x^2$$

$$2x-4 = 0 \qquad\qquad \to \qquad\qquad x = 2$$

A sketch of the region is to the right and we can see that we'll need two integrals to find the area. Here they are,



$$A = \int_{-3}^2 x^2-\left(x^2+2x-4\right)dx + \int_2^3 x^2+2x-4-\left(x^2\right)dx$$

$$= \int_{-3}^2 4-2x\,dx + \int_2^3 2x-4\,dx = \left(4x-x^2\right)\Big|_{-3}^2 + \left(x^2-4x\right)\Big|_2^3 = 25+1 = \boxed{26}$$

8. (2 pts) First we'll need intersection points.

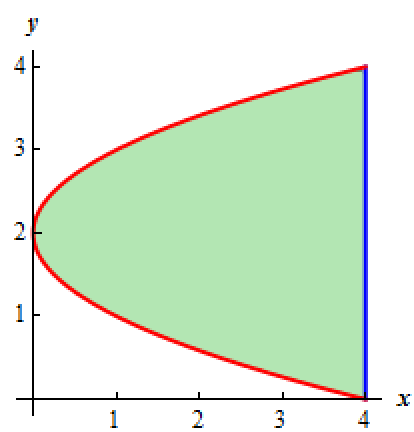
$$(y-2)^2 = 4$$

$$y-2 = \pm 2 \qquad\qquad \to \qquad\qquad y=0,\; y=4$$

A sketch of the region is to the right. Here is the integral.

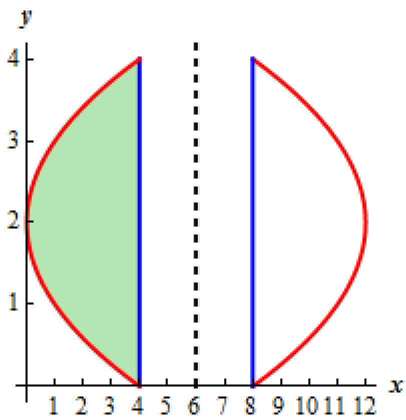
$$A = \int_0^4 4-(y-2)^2\,dy = \int_0^4 4dy - \int_0^4 (y-2)^2\,dy \qquad u = y-2$$

$$= \int_0^4 4dy - \int_{-2}^2 u^2 du = 4y\Big|_0^4 - \tfrac{1}{3}u^3\Big|_{-2}^2 = 16-\left(\tfrac{8}{3}+\tfrac{8}{3}\right) = \boxed{\tfrac{32}{3}}$$



10. (2 pts) A sketch of the region (shaded) and the mirrored region is to the right. The intersection points are clearly at $y=0$ and $y=4$. The radii are then,

$$\text{o.r.}=6-(y-2)^2 \qquad \text{i.r.}=6-4=2$$



The volume is then,

$$\begin{aligned} V &= \int_0^4 \pi\left[\left(6-(y-2)^2\right)^2-(2)^2\right]dy \\ &= \pi\int_0^4 y^4-8y^3+12y^2+16y\,dy \\ &= \pi\left(\tfrac{1}{5}y^5-2y^4+4y^3+8y^2\right)\Big|_0^4=\boxed{\tfrac{384}{5}\pi} \end{aligned}$$

Not Graded

2.

$$u=1+4\mathbf{e}^x \quad du=4\mathbf{e}^x dx \quad \mathbf{e}^x dx=\tfrac{1}{4}du \qquad x=\ln(2): u=1+4(2)=9,\ x=\ln(5): u=21$$

$$\int_{\ln(2)}^{\ln(5)}\frac{\mathbf{e}^x}{1+4\mathbf{e}^x}dx=\tfrac{1}{4}\int_9^{21}\frac{1}{u}du=\tfrac{1}{4}\ln|u|\Big|_9^{21}=\boxed{\tfrac{1}{4}\ln(21)-\tfrac{1}{4}\ln(9)=0.22182}$$

4. I had intended for the denominator of the first term to be $4y-8$ in which case the answer would be the following…. For this integral notice that we will have division by zero issues in the first term at $y=2$ and this is inside the interval of integration and so **this integral can't be done**.

However, since I messed up the problem statement here is how to work the problem as written. Note that because of the *y* in the numerator we'll need to do one of those "tricky" substitutions for the first term.

$$\int_1^6\frac{3y}{8y-4}-\cos(y)\mathbf{e}^{\sin(y)}\,dy=\int_1^6\frac{3y}{8y-4}dy-\int_1^6\cos(y)\mathbf{e}^{\sin(y)}\,dy$$

$$u=8y-4 \quad du=8dy \quad \tfrac{1}{8}du=dy \qquad y=\tfrac{1}{8}(u+4) \qquad y=1:\ u=4,\quad y=6:\ u=44$$

$$v=\sin(y) \quad dy=\cos(y)dy \quad y=1: v=\sin(1), \quad y=6: v=\cos(6)$$

$$\begin{aligned} \int_1^6\frac{3y}{8y-4}-\cos(y)\mathbf{e}^{\sin(y)}\,dy &= \tfrac{1}{8}\int_1^6\frac{\tfrac{3}{8}(u+4)}{u}du-\int_{\sin(1)}^{\sin(6)}\mathbf{e}^v\,dv=\tfrac{3}{64}\int_1^6 1+\frac{4}{u}du-\int_{\sin(1)}^{\sin(6)}\mathbf{e}^v\,dv \\ &= \tfrac{3}{64}\left(u+4\ln|u|\right)\Big|_1^6-\mathbf{e}^v\Big|_{\sin(1)}^{\sin(6)}=\boxed{\tfrac{3}{64}\left(5+4\ln(6)\right)-\mathbf{e}^{\sin(1)}+\mathbf{e}^{\sin(6)}} \end{aligned}$$

Overall this was a good problem as it covered a lot of things that we talked about in a single problem.

5. We'll need to split this integral up as both terms require different substitutions.

$$\int_2^1 \sqrt{1+2x} + (1-x)^5\,dx = \int_2^1 \sqrt{1+2x}\,dx + \int_2^1 (1-x)^5\,dx$$

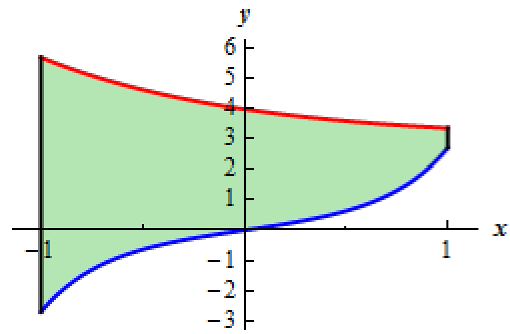
$$u = 1+2x \quad du = 2dx \quad dx = \tfrac{1}{2}du \qquad x=2: u=5, \quad x=1: u=3$$

$$v = 1-x \quad dv = -dx \quad dx = -dv \qquad x=2: v=-1, \quad x=1: v=0$$

$$\int_2^1 \sqrt{1+2x} + (1-x)^5\,dx = \tfrac{1}{2}\int_5^3 u^{\frac{1}{2}}\,du - \int_{-1}^0 v^5\,dv = \tfrac{1}{3}u^{\frac{3}{2}}\Big|_5^3 - \tfrac{1}{6}v^6\Big|_{-1}^0 = \boxed{\tfrac{1}{3}\sqrt{3} - \tfrac{1}{3}\sqrt{5} + \tfrac{1}{6}}$$

6. A sketch of the region is to the right ($y = x\mathbf{e}^{x^2}$ is the lower curve). Here is the integral.

$$A = \int_{-1}^1 3 + \mathbf{e}^{-x} - x\mathbf{e}^{x^2}\,dx = \int_{-1}^1 3\,dx + \int_{-1}^1 \mathbf{e}^{-x}\,dx - \int_{-1}^1 x\mathbf{e}^{x^2}\,dx$$



As noted the integral needs split into three separate integrals. The second and third each need a substitution and each substitution is different.

$$u = -x \quad du = -dx \quad dx = -du \qquad x=-1: u=1, \quad x=1: u=-1$$

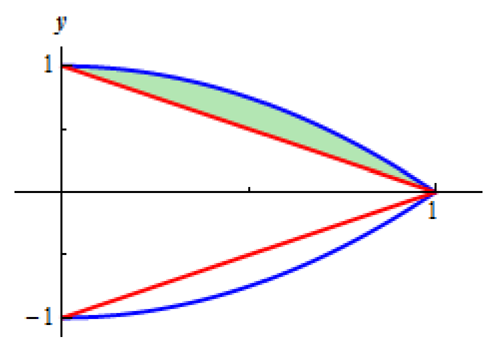
$$v = x^2 \quad dv = 2xdx \quad xdx = \tfrac{1}{2}du \qquad x=-1: u=1, \quad x=1: u=1$$

$$A = \int_{-1}^1 3 + \mathbf{e}^{-x} - x\mathbf{e}^{x^2}\,dx = \int_{-1}^1 3\,dx - \int_1^{-1} \mathbf{e}^u\,du - \tfrac{1}{2}\int_1^1 \mathbf{e}^v\,dv = 3x\Big|_{-1}^1 - \mathbf{e}^u\Big|_1^{-1} + 0 = \boxed{6 + \mathbf{e} - \mathbf{e}^{-1}}$$

Note that we know that the third integral will be zero because the upper and lower limits are the same.

9. A sketch of the region (shaded) and the mirrored region is to the right. The intersection points are clearly at $x = 0$ and $x = 1$. The radii are then,

$$\text{o.r.} = 1 - x^2 \qquad \text{i.r.} = 1 - x$$



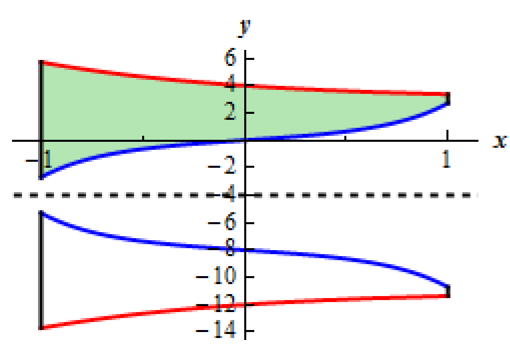
The volume is then,

$$V = \int_0^1 \pi\left[\left(1-x^2\right)^2 - (1-x)^2\right]dx = \pi\int_0^1 x^4 - 3x^2 + 2x\,dx$$

$$= \pi\left(\tfrac{1}{5}x^5 - x^3 + x^2\right)\Big|_0^1 = \boxed{\tfrac{1}{5}\pi}$$

11. A sketch of the region (shaded) and the mirrored region is to the right. The radii are,

$$\text{o.r.} = 4 + 3 + \mathbf{e}^{-x} = 7 + \mathbf{e}^{-x} \qquad \text{i.r.} = 4 + x\mathbf{e}^{x^2}$$



The integral giving the volume is then,

$$V = \int_0^1 \pi\left[\left(7+\mathbf{e}^{-x}\right)^2 - \left(4 + x\mathbf{e}^{x^2}\right)^2\right]dx$$

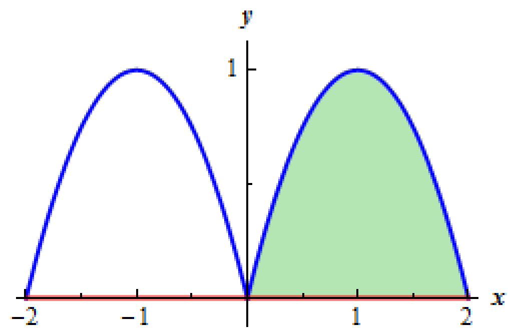
Note that you do not have the knowledge to completely evaluate this integral at this point. Hence the set up only instruction……

12. A sketch of the region (shaded) and the mirrored region is to the right. The radius and height are are,

$$\text{radius} = x \qquad \text{height} = 2x - x^2$$

The volume is then,

$$V = \int_0^2 2\pi x\left(2x - x^2\right)dx = 2\pi\int_0^2 2x^2 - x^3\,dx$$
$$= 2\pi\left(\tfrac{2}{3}x^3 - \tfrac{1}{4}x^4\right)\Big|_0^2 = \boxed{\tfrac{8}{3}\pi}$$

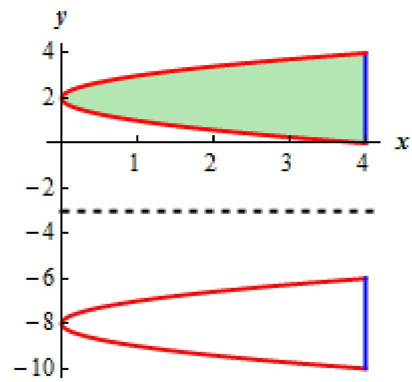


13. A sketch of the region (shaded) and the mirrored region is to the right. The radius and width are are,

$$\text{radius} = y + 3 \qquad \text{width} = 4 - \left(y - 2\right)^2$$

The volume is then,

$$V = \int_0^4 2\pi\left(y+3\right)\left(4 - \left(y-2\right)^2\right)dy$$
$$= 2\pi\int_0^4 12y + y^2 - y^3\,dy$$
$$= 2\pi\left(6y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4\right)\Big|_0^4 = \boxed{\tfrac{320}{3}\pi}$$



14. A sketch of the region (shaded) and the mirrored region is to the right. The radius and height are are,

$$\text{radius} = 2 - x \qquad \text{height} = 3 + \mathbf{e}^{-x} - x\mathbf{e}^{x^2}$$

The integral giving the volume is then,

$$V = \int_{-1}^{1} 2\pi\left(2 - x\right)\left(3 + \mathbf{e}^{-x} - x\mathbf{e}^{x^2}\right)dx$$

Note that you do not have the knowledge to completely evaluate this integral at this point. Hence the set up only instruction……