## Set 4 – Solutions

**1. (2 pts)** First :  $F(x, y, z) = x(z^2 - y) - e^{x+3y} = 107$ . Now all we need to do is a little gradient work.

$$\nabla F = \left\langle z^2 - y - \mathbf{e}^{x+3y}, -x - 3\mathbf{e}^{x+3y}, 2xz \right\rangle \qquad \nabla F \left( 6, -2, 4 \right) = \left\langle 17, -9, 48 \right\rangle$$
  
The tangent plane is :  $17 \left( x - 6 \right) - 9 \left( y + 2 \right) + 48 \left( z - 4 \right) = 0 \qquad \Rightarrow \qquad \boxed{17x - 9y + 48z = 312}$   
The normal line is :  $\overrightarrow{r} \left( t \right) = \left\langle 6, -2, 4 \right\rangle + t \left\langle 17, -9, 48 \right\rangle$ 

## 3. (2 pts) Here are all the derivatives and D.

$$h_x = 4x^3 - 2y^2 - 16x \quad h_y = -4xy + 12y \qquad h_{xx} = 12x^2 - 16$$
$$h_{xy} = -4y \qquad h_{yy} = -4x + 12 \qquad D = (12x^2 - 16)(-4x + 12) - 16y^2$$

Now, find the critical points.

$$4x^{3} - 2y^{2} - 16x = 0$$
  

$$-4xy + 12y = 4y(3 - x) = 0 \implies y = 0, x = 3$$
  

$$y = 0: 4x^{3} - 16x = 4x(x^{2} - 4) = 0 \implies x = 0, x = \pm 2$$
  

$$x = 3: 60 - 2y^{2} = 0 \implies y = \pm\sqrt{30}$$

So, it looks like we've got 5 critical points. Here they are and their classifications.

$$(0,0)$$
: $D = -192 < 0$ Saddle Point $(2,0)$ : $D = 128 > 0$  $h_{xx}(2,0) = 32 > 0$ Relative Minimum $(-2,0)$ : $D = 640 > 0$  $h_{xx}(-2,0) = 32 > 0$ Relative Minimum $(3,\sqrt{30})$ : $D = -480 < 0$ Saddle Point $(3,-\sqrt{30})$ : $D = -480 < 0$ Saddle Point

5. (3 pts) First we need the critical points.

$$f_x = 12 + 4xy + 4y^2$$
  

$$f_y = 2x^2 + 8xy = 2x(x+4y) = 0 \quad \rightarrow \quad x = 0, x = -4y$$
  

$$x = 0:12 + 4y^2 = 0 \quad \times$$
  

$$x = -4y:12 - 12y^2 = 0 \quad \rightarrow \quad y = \pm 1$$

So, it looks like we have two critical points : (-4,1) and (4,-1).

Now, let's deal with the boundary. From a quick sketch of D we see that the three sides are given by,

Upper : y = 2x Left : x = 6 Bottom :  $y = -\frac{1}{2}x$ 

From the sketch we can see that (-4,1) can't possibly be in the region. Likewise at x = 4 the upper/lower equations gives 8 and -2 and we can clearly see that -1 falls in this range and so (4,-1) will be in the region and so we'll keep that around.

Now, let's run through each edge of the region.

**Upper**:  $h(x) = f(x, 2x) = 12x + 20x^3 \rightarrow h'(x) = 12 + 60x^2 \rightarrow \text{No critical points}$ From the upper side we have only the end points : (0,0) and (6,12).

Left:  $h(y) = f(6, y) = 72 + 72y + 24y^2 \rightarrow h'(y) = 72 + 48y \rightarrow y = -\frac{3}{2}$ 

Here we have the critical point  $(6, -\frac{3}{2})$  and the end points : (6,12) and (6,-3).

**Bottom** :  $h(x) = f(x, -\frac{1}{2}x) = 12x \rightarrow h'(x) = 12 \rightarrow$  No critical points From the bottom side we have only the end points : (0,0) and (6,-3).

Now all we need to do is evaluate the function at these points.

f(4,-1) = 32  $f(6,-\frac{3}{2}) = 18$  f(0,0) = 0 f(6,12) = 4392 f(6,-3) = 72

So, the absolute maximum is 4392 at (6,12) and the absolute minimum is 0 at (0,0).

**8. (3 pts)** From the constraint we can see that we must have  $-5 \le x, y, z \le 5$  and so we are on a bounded region and are therefore guaranteed to have absolute extrema.

Here are the equations we need to solve and notice that we can't have  $\lambda = 0$  as the equations two and three would clearly not be valid.

$3 = 2x\lambda$	$\Rightarrow$	$x = \frac{3}{2\lambda}$
$-4 = 2 y \lambda$	$\Rightarrow$	$y = -\frac{2}{\lambda}$
$-2z = 2z\lambda$	$\Rightarrow$	$z = 0$ or $\lambda = -1$
$x^2 + y^2 + z^2 = 25$		

The third equation gives us two possibilities. Note however, that regardless of which of those we use the results from the first two equations will be true regardless. So, let's go through the two possibilities from the third equation.

$$z = 0 : \left(\frac{3}{2\lambda}\right)^2 + \left(-\frac{2}{\lambda}\right)^2 = \frac{25}{4\lambda^2} = 25 \qquad \Rightarrow \qquad \lambda = \pm \frac{1}{2}$$

This case yields two points,

$$(3,-4,0)$$
  $(-3,4,0)$ 

$$\lambda = -1 : x = -\frac{3}{2}, y = 2 \rightarrow \frac{9}{4} + 4 + z^2 = \frac{25}{4} + z^2 = 25 \qquad \Rightarrow \qquad z = \pm \sqrt{\frac{75}{4}} = \pm \frac{5\sqrt{3}}{2}$$
  
This case yields two points,

$$\left(-\frac{3}{2}, 2, \frac{5\sqrt{3}}{2}\right)$$
  $\left(-\frac{3}{2}, 2, -\frac{5\sqrt{3}}{2}\right)$ 

Plugging these four points into the function gives,

$$f(3,-4,0) = 25 \qquad f(-3,4,0) = -25 \qquad f\left(-\frac{3}{2},2,\frac{5\sqrt{3}}{2}\right) = f\left(-\frac{3}{2},2,-\frac{5\sqrt{3}}{2}\right) = -\frac{125}{4}$$

The minimum is  $-\frac{125}{4} = -31.25$  which occurs at  $\left(-\frac{3}{2}, 2, \frac{5\sqrt{3}}{2}\right)$  and  $\left(-\frac{3}{2}, 2, -\frac{5\sqrt{3}}{2}\right)$ . The maximum is 25 which occurs at (3, -4, 0).

## Not Graded

**2.** The gradient of the function  $F(x, y, z) = x^2 + 6y^2 - 3z^2 = -6$  is  $\nabla F = \langle 2x, 12y, -6z \rangle$  and we know that this is orthogonal to the surface at any point and so is the normal vector for the tangent plane. We want the point(s) where this is parallel to the normal vector of the given plane,  $\vec{n} = \langle 2, -1, -4 \rangle$ . We know that parallel vectors will be parallel if they are scalar multiplies of each other. In other words, there is a number *c* so that,

$$\nabla F = c\vec{n}$$
  $\Rightarrow$   $\langle 2x, 12y, -6z \rangle = c \langle 2, -1, -4 \rangle = \langle 2c, -c, -4c \rangle$ 

Setting components equal gives the following three equations that can be solved for x, y, and z.

$$2x = 2c x = c$$

$$12y = -c \Rightarrow y = -\frac{1}{12}c$$

$$-6z = -4c z = \frac{2}{3}c$$

Now, since the points must be on the surface they must also satisfy  $x^2 + 6y^2 - 3z^2 = -6$  so the equations above into this and solve for *c*.

$$(c)^{2} + 6(-\frac{1}{12}c)^{2} - 3(\frac{2}{3}c)^{2} = -6 \qquad \Rightarrow \qquad -\frac{7}{24}c^{2} = -6 \qquad \Rightarrow \qquad c = \pm\sqrt{\frac{144}{7}} = \pm\frac{12}{\sqrt{7}}$$

We've got 2 c's and so that means that we have two possible points. One for each c. The points are,

Math 2415

$$\left(\frac{12}{\sqrt{7}}, -\frac{1}{\sqrt{7}}, -\frac{8}{\sqrt{7}}\right) \qquad \left(-\frac{12}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{8}{\sqrt{7}}\right)$$

**4.** Here are all the derivatives and *D*.

$$g_{x} = \frac{3x^{2} - 8x - 2}{y^{2} + 10}$$

$$g_{y} = -\frac{\left(x^{3} - 4x^{2} - 2x\right)\left(2y\right)}{\left(y^{2} + 10\right)^{2}}$$

$$g_{xx} = \frac{6x - 8}{y^{2} + 10}$$

$$g_{yy} = \frac{\left(x^{3} - 4x^{2} - 2x\right)\left(6y^{2} - 20\right)}{\left(y^{2} + 10\right)^{3}}$$

$$g_{xy} = -\frac{\left(3x^{2} - 8x - 2\right)\left(2y\right)}{\left(y^{2} + 10\right)^{2}}$$

$$D = \frac{\left(x^{3} - 4x^{2} - 2x\right)\left(6x - 8\right)\left(6y^{2} - 20\right)}{\left(y^{2} + 10\right)^{3}} - \frac{4y^{2}\left(3x^{2} - 8x - 2\right)^{2}}{\left(y^{2} + 10\right)^{4}}$$

Now, find the critical points. I'll leave it to you to verify that the denominator will never be zero for any real numbers. Therefore, in order for the two first derivatives to be zero the numerators will have to be zero. So, setting those equal to zero and solving gives,

$$2yx(x^{2}-4x-2) = 0 \qquad \Rightarrow \qquad y = 0, \ x = 0, \ x = 2 \pm \sqrt{6}$$
$$3x^{2}-8x-2 = 0 \qquad \Rightarrow \qquad x = \frac{4\pm\sqrt{22}}{3} = -0.2301, \ 2.8968$$

Okay, we need to make sense of all of this. Recall that critical points will be points where BOTH of the first derivatives will be zero. From the second we know that  $g_y$  will ONLY be zero for the two given values of x. This means that the three value of x we got from the first equation will NOT be part of any critical points because  $g_y \neq 0$  for these values of x. This means that we are left with y = 0 from the first equation and so the two critical points are,

$$(-0.2301,0)$$
  $(2.8968,0)$ 

Finally classify them and note that because of y = 0 in the critical points the evaluation of *D* won't be as bad as it first looked like it might be.

$$(-0.2301,0)$$
 :  $D = 0.0044 > 0$   $g_{xx}(-0.2301,0) = -0.9381 < 0$  Relative Maximum  
(2.8968,0) :  $D = 0.2824 > 0$   $g_{xx}(2.8968,0) = 0.9381 > 0$  Relative Minimum

**6.** From the constraint we can see that we must have  $-2 \le x, y \le 2$  and so we are on a bounded region and are therefore guaranteed to have absolute extrema.

Set 4 – Solutions

Here are the equations we need to solve and note that  $\lambda \neq 0$  as this would give x and y both zero, which can't happen.

$$8x = 4\lambda x^{3} \qquad \Rightarrow \qquad x = 0 \quad or \quad x^{2} = \frac{2}{\lambda}$$
$$-6y = 4\lambda y^{3} \qquad \Rightarrow \qquad y = 0 \quad or \quad y^{2} = -\frac{3}{2\lambda}$$
$$x^{4} + y^{4} = 16$$

Note that if we use both of the conditions  $x^2 = \frac{2}{\lambda}$  and  $y^2 = -\frac{3}{2\lambda}$  together we will get complex solutions. If  $\lambda > 0$  then *x* will be real, but *y* will be complex. Likewise, if  $\lambda < 0$  then *y* will be real and *x* will be complex.

So, this leaves either x = 0 or y = 0. So, plug each of these into the constraint and note that they can't both be zero at the same time! Note as well that we won't need the other two solutions here as those will simply tell us what  $\lambda$  would be and we don't need that here.

 $x=0: y^4=16 \implies y=\pm 2 \implies (0,2) \text{ and } (0,-2)$ 

y=0:  $x^4=16 \implies x=\pm 2 \implies (2,0)$  and (-2,0)

To finish the problem out we just need to evaluate the function as these points.

$$f(0,\pm 2) = -12$$
  $f(\pm 2,0) = 16$ 

So, the absolute maximum is 16 which occurs at (0,2) and (0,-2) while the absolute minimum is -12 which occurs at (2,0) and (-2,0).

7. For this problem we need to assume that  $x \le 0$  in order to make sure that solutions will in fact exist. If we allowed any x then we could take x as large and positive as we wanted and with sufficiently large y and/or z we'd be able to meet the constraint. This however would allow the function to grow as large as we wanted. By restricting  $x \le 0$  force all three terms in the constraint to be positive or zero and because the sum of the three positive (or zero) terms must be 32 then neither can be too large and so we know that we will have a minimum and (more importantly) a maximum value of the function.

So, let's go through the problem.

$$yz = -2\lambda \qquad \Rightarrow \qquad xyz = -2x\lambda$$
$$xz = 2y\lambda \qquad \Rightarrow \qquad xyz = 2y^{2}\lambda$$
$$xy = 4z\lambda \qquad \Rightarrow \qquad xyz = 4z^{2}\lambda$$
$$y^{2} + 2z^{2} - 2x = 32$$

We got the second set of equations by multiplying the first by *x*, the second by *y* and the third by *z*. If we set the first and second equal as well as the second and third equal we get,

$$2\lambda(y^{2} + x) = 0 \qquad \Rightarrow \qquad \lambda = 0 \quad \text{or} \quad x = -y^{2}$$
$$2\lambda(y^{2} - 2z^{2}) = 0 \qquad \Rightarrow \qquad \lambda = 0 \quad \text{or} \quad y^{2} = 2z^{2}$$

Let's start off by assuming that  $\lambda = 0$ . In this case the three original equations become,

yz = 0	$\Rightarrow$	y = 0	or	z = 0
xz = 0	$\Rightarrow$	x = 0	or	z = 0
xy = 0	$\Rightarrow$	x = 0	or	y = 0

We can't have all three zero since that won't satisfy the constraint. However, notice that if y=0 then we must have either x or z be zero in order to satisfy the second equation. Likewise if x=0 then either y or z must be zero to satisfy the first equation. Finally, if z=0 then either x or y must be zero in order to satisfy the third equation. So, we can't have all three be zero and we can't have only one be zero. However, in this case, we can have two of them be zero. So, if we assume that two are zero and plug these into the constraint to solve for the third we get the following five points,

$$(0,0,\pm 4)$$
  $(0,\pm\sqrt{32},0)$   $(-16,0,0)$ 

Now, let's assume that  $\lambda \neq 0$ . This forces  $y^2 = 2z^2$  and  $x = -y^2 = -2z^2$  Plugging these into the constraint gives,

$$2z^2 + 2z^2 + 2z^2 = 8z^2 = 32 \qquad \Rightarrow \qquad z = \pm 2$$

In either case for x we get that x = -8 and  $y = \pm \sqrt{8}$  and so we get the following set of points.

$$(-8,\sqrt{8},2)$$
  $(-8,-\sqrt{8},2)$   $(-8,\sqrt{8},-2)$   $(-8,-\sqrt{8},-2)$ 

Finally all we need to do is plug into the function.

$$f(0,0,\pm 4) = f(0,\pm\sqrt{32},0) = f(-16,0,0) = 0$$
$$f(-8,-\sqrt{8},2) = f(-8,\sqrt{8},-2) = 16\sqrt{8}$$
$$f(-8,\sqrt{8},2) = f(-8,-\sqrt{8},-2) = -16\sqrt{8}$$

The maximum value is then  $16\sqrt{8}$  which occurs at  $\left(-8, -\sqrt{8}, 2\right)$  and  $\left(-8, \sqrt{8}, -2\right)$ . The minimum value is  $-16\sqrt{8}$  which occurs at  $\left(-8, \sqrt{8}, 2\right)$  and  $\left(-8, -\sqrt{8}, -2\right)$ .