Math 2415 **Homework Set 9 – Solutions** **10 Points**

1. (2 pts) Here's the parameterization of the curve and we'll also need the derivative of the parameterization.

$$\vec{r}\left(t\right)=\left\langle 4\cos t,3\sin t\right\rangle,\ \ \tfrac{1}{2}\pi\le t\le\tfrac{3}{2}\pi \qquad\qquad \vec{r}'\left(t\right)=\left\langle -4\sin t,3\cos t\right\rangle$$

Next, we'll need the dot product,

$$\begin{aligned}\vec{F}\left(\vec{r}\left(t\right)\right)\bullet\vec{r}'\left(t\right)&=\left\langle 12\cos t\sin t,3\sin t-1\right\rangle\bullet\left\langle -4\sin t,3\cos t\right\rangle\\&=-48\cos t\sin^2 t+9\sin t\cos t-3\cos t\end{aligned}$$

The integral is then,

$$\int\limits_C \vec{F}\bullet d\vec{r}=\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}-48\cos t\sin^2 t+\tfrac{9}{2}\sin\left(2t\right)-3\cos t\,dt=\boxed{38}$$

4. (2 pts) This is not as difficult as it might seem at first glance. One way to do the problem is to go back and redo the integral from **#1** only this time we need to do is on the interval $0\le t\le 2\pi$, *i.e.* one complete revolution.

$$\int\limits_C \vec{F}\bullet d\vec{r}=\int_0^{2\pi}-48\cos t\sin^2 t+\tfrac{9}{2}\sin\left(2t\right)-3\cos t\,dt=\boxed{0}$$

So, in this case this doesn't do us any good. If we'd gotten a non-zero value we'd know it was not independent of path because the integral over every closed path must be zero. However, just because we got zero here doesn't mean anything. It may be independent of path or it may not be. So, let's look at another, and easier way, is to go back and verify that the vector field is not conservative ($P_y=x\ne 0=Q_x$) and we know that if an integral is independent of path it MUST be conservative. So, because the vector field is not conservative the integral **can't be independent of path**.

9. (3 pts) Because we don't know how to verify this is conservative (yet) we'll have to believe that it is. Here are the various parts of the vector field.

$$P=2x\ln\left(y^2z\right)\qquad Q=\frac{2x^2}{y}-27y^2z^4\qquad R=\frac{x^2}{z}-36y^3z^3$$

Integrate with respect to *x* first

$$f\left(x,y,z\right)=\int 2x\ln\left(y^2z\right)dx=x^2\ln\left(y^2z\right)+g\left(y,z\right)$$

Differentiating with respect to *y* and setting equal to *Q* gives,

$$f_y=\frac{2x^2}{y}+g_y\left(y,z\right)=\frac{2x^2}{y}-27y^2z^4\quad\Rightarrow\quad g_y\left(y,z\right)=-27y^2z^4$$

$$g\left(y,z\right)=-9y^3z^4+h\left(z\right)$$

The potential function is now,

$$f\left(x,y,z\right)=x^2\ln\left(y^2z\right)-9y^3z^4+h\left(z\right)$$

Differentiating with respect to *z* and setting equal to *R* gives,

$$f_z = \frac{x^2}{z} - 36y^3z^3 + h'(z) = \frac{x^2}{z} - 36y^3z^3 \qquad \Rightarrow \qquad h'(z) = 0 \qquad \Rightarrow \qquad h(z) = c$$

The potential function is then,

$$f(x,y,z) = x^2 \ln\left(y^2 z\right) - 9y^3z^4 + c$$

Finally, the integral (ignoring the *c* because it will cancel),

$$\int_C \vec{F} \bullet d\vec{r} = f\left(\vec{r}(2)\right) - f\left(r(1)\right) = f(0,8,1) - f(-4,2,1) = \boxed{-4536 - 16\ln 4 = -4558.18}$$

10. (3 pts)

(a) Here's a parameterization of the curve.

$$\vec{r}(t) = \langle 3\cos t, 3\sin t\rangle, \;\; 0 \le t \le 2\pi \qquad\qquad \vec{r}'(t) = \langle -3\sin t, 3\cos t\rangle$$

The integral is then,

$$\oint_C (2xy + 3y)\,dx - \left(3x - x^2\right)dy = \int_0^{2\pi} -3\left(18\cos t\sin^2 t + 9\sin^2 t\right) - 3\left(9\cos^2 t - 9\cos^3 t\right)dt$$

$$= \int_0^{2\pi} -54\cos t\sin^2 t + 27\cos^3 t - 27\,dt = \boxed{-54\pi}$$

(b) Here's the integral after applying Green's Theorem.

$$\oint_C (2xy + 3y)\,dx - \left(3x - x^2\right)dy = \iint_D 2x - 3 - (3 + 2x)\,dA = -6\iint_D dA = -6(9\pi) = \boxed{-54\pi}$$

Note that we didn't need to actually do the integral since it's just the area of the disk.

Not Graded

2. We'll need the derivative of the parameterization and the dot product.

$$\vec{r}'(t) = -\vec{i} + 3t^2\vec{j} - 18t^2\vec{k}$$

$$\vec{F}\left(\vec{r}(t)\right)\bullet\vec{r}(t) = \left(-6t^6\,\vec{i} - \left(9 - t + 36t^6\right)\vec{j} + \cos\left(4t^3\right)\vec{k}\right)\bullet\left(-\vec{i} + 3t^2\vec{j} - 18t^2\vec{k}\right)$$

$$= 6t^6 - 3t^2\left(9 - t + 36t^6\right) - 18t^2\cos\left(4t^3\right) = -108t^8 + 6t^6 + 3t^3 - 27t^2 - 18t^2\cos\left(4t^3\right)$$

The integral is then,

$$\int_C \vec{F}\bullet d\vec{r} = \int_0^1 -108t^8 + 6t^6 + 3t^3 - 27t^2 - 18t^2\cos\left(4t^3\right)dt = \boxed{-\tfrac{543}{28} - \tfrac{3}{2}\sin(12) = -18.2577}$$

3. Not much here other than to use the Fundamental Theorem of Calculus.

$$\int_C \nabla f \bullet d\vec{r} = f\left(\vec{r}(3)\right) - f\left(\vec{r}(-1)\right) = f(12,60,63) - f(0,4,7) = (90720) - (0) = \boxed{90676.0081}$$

5. First run through the test.

$$P = 8xy^3 + 3x^2y \qquad P_y = 24xy^2 + 3x^2$$
$$Q = 12x^2y^2 - x^3 \qquad Q_x = 24xy^2 - 3x^2$$

So, $P_y \neq Q_x$ and so the vector field is **NOT** conservative.

6. First run through the test.

$$P = 6x + 3x^2y^4 \qquad P_y = 12x^2y^3$$
$$Q = -\left(8y - 4x^3y^3 + 7\right) \qquad Q_x = 12x^2y^3$$

So, the vector field is conservative. We'll start off by integrating with respect to *x* first.

$$f\left(x,y\right) = \int P\,dx = \int 6x + 3x^2y^4\,dx = 3x^2 + x^3y^4 + h\left(y\right)$$

Differentiate with respect to *y* and set equal to *Q*.

$$f_y = 4x^3y^3 + h'\left(y\right) = -8y + 4x^3y^3 - 7 \;\Rightarrow\; h'\left(y\right) = -8y - 7 \Rightarrow \; h\left(y\right) = -4y^2 - 7y + c$$

The potential function is then : $\boxed{f\left(x,y\right) = 3x^2 + x^3y^4 - 4y^2 - 7y + c}$

7. We already found the potential function for this in **#6** and so all we need to do is compute the integral (ignoring the *c* since it will cancel out).

$$\int_C \vec{F} \bullet d\vec{r} = f\left(4,0\right) - f\left(1,-3\right) = 48 - \left(69\right) = \boxed{-21}$$

8. We'll assume that this is conservative as we're being asked to find the potential function. So,

$$P = -4x - \mathbf{e}^{2y}\cos\left(4y - x\right) \qquad Q = 1 + 2\mathbf{e}^{2y}\sin\left(4y - x\right) + 4\mathbf{e}^{2y}\cos\left(4y - x\right)$$

We'll integrate with respect to *x* first.

$$f\left(x,y\right) = \int P\,dx = \int -4x - \mathbf{e}^{2y}\cos\left(4y - x\right)dx = -2x^2 + \mathbf{e}^{2y}\sin\left(4y - x\right) + h\left(y\right)$$

Differentiate with respect to *y* and set equal to *Q*.

$$f_x\left(x,y\right) = 2\mathbf{e}^{2y}\sin\left(4y - x\right) + 4\mathbf{e}^{2y}\cos\left(4y - x\right) + h'\left(y\right) = 1 + 2\mathbf{e}^{2y}\sin\left(4y - x\right) + 4\mathbf{e}^{2y}\cos\left(4y - x\right)$$
$$h'\left(y\right) = 1 \qquad \Rightarrow \qquad h\left(y\right) = y + c$$

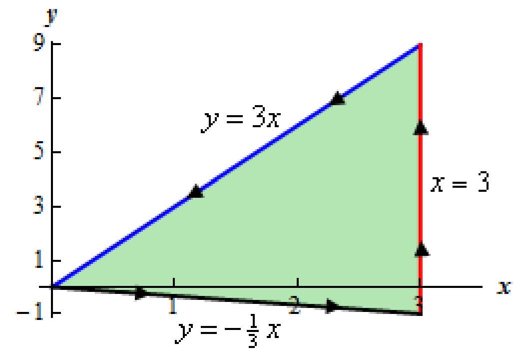
The potential function is then,

$$f\left(x,y\right) = -2x^2 + \mathbf{e}^{2y}\sin\left(4y - x\right) + y + c$$

In order to evaluate the integral we need the initial and final point of *C* and because we know the center and radius of the circle we can get these easily enough : Initial Point : (3, 2), Final Point : (-5, 2). The integral is then (ignoring the *c* because it will cancel),

$$\int_C \vec{F} \bullet d\vec{r} = f(-5,2) - f(3,2) = \boxed{-32 + \mathbf{e}^4 \left(\sin(13) - \sin(5)\right) = 43.2958}$$

11. (a) A sketch of the curve *C* and the enclosed region *D* is to the right. Here are the parameterizations for each curve and because we need to pay attention to direction we can can't use the equation for the upper line.



$$C_1 : \vec{r}(t) = \left\langle t, -\tfrac{1}{3}t \right\rangle,\ 0 \le t \le 3$$
$$C_2 : \vec{r}(t) = \left\langle 3, t \right\rangle,\ -1 \le t \le 9$$
$$C_3 : \vec{r}(t) = \left\langle 3-3t, 9-9t \right\rangle,\ 0 \le t \le 1$$

Here's the integral done directly for each curve.

$$\oint_{C_1} x^2\,dy - (4y - 8xy)\,dx = \int_0^3 t^2\left(-\tfrac{1}{3}\right) - \left(-\tfrac{4}{3}t + \tfrac{8}{3}t^2\right)(1)\,dt = \int_0^3 \tfrac{4}{3}t - 3t^2\,dt = \underline{-21}$$

$$\oint_{C_2} x^2\,dy - (4y - 8xy)\,dx = \int_{-1}^9 (9)(1) - \left(4(t) - 8(3)(t)\right)(0)\,dt = \int_{-1}^9 9\,dt = \underline{90}$$

$$\oint_{C_3} x^2\,dy - (4y - 8xy)\,dx = \int_0^1 (3-3t)^2(-9) - \left(4(9-9t) - 8(3-3t)(9-9t)\right)(-3)\,dt$$
$$= \int_0^1 -729t^2 + 1350t - 621\,dt = \underline{-189}$$

The integral is then,

$$\oint_C x^2\,dy - (4y - 8xy)\,dx = -21 + 90 - 189 = \boxed{-120}$$

(b) Using Green's Theorem the integral is,

$$\oint_C x^2\,dy - (4y - 8xy)\,dx = \iint_D 2x - \left[-(4 - 8x)\right]dA = \iint_D 4 - 6x\,dA$$

We'll use the following for limits for *D* : $0 \le x \le 3,\ -\tfrac{1}{3}x \le y \le 3x$. The integral is then,

$$\oint_C x^2\,dy - (4y - 8xy)\,dx = \int_0^3 \int_{-\frac{1}{3}x}^{3x} 4 - 6x\,dy\,dx = \tfrac{1}{3}\int_0^3 40x - 80x^2\,dx = \boxed{-120}$$