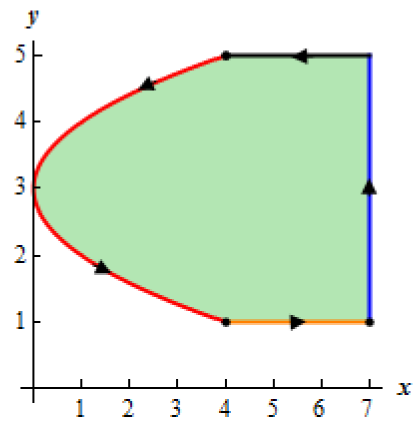
**1.** A sketch of the curve/region is to the right. The limits that we'll use for the double integral are : $1 \le y \le 5,\ 7 \le x \le (y-3)^2$




The integral is,

$$\oint_C 5(1-4xy)\,dx + 9x\,dy = \iint_D 9+20x\,dA$$

$$= \int_1^5 \int_7^{(y-3)^2} 9+20x\,dx\,dy$$

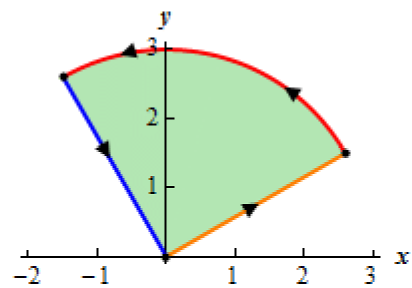
$$= \int_1^5 9(y-3)^2 + 10(y-3)^4 - 553\,dx$$

$$= \left(3(y-3)^3 + 2(y-3)^5 + 476y\right)\Big|_1^5 = \boxed{-2036}$$

**2.** A sketch of the curve/region is to the right. The limits are,

$$0 \le r \le 3, \qquad \tfrac{\pi}{6} \le \theta \le \tfrac{2\pi}{3}$$

Note that for the angles we can use the endpoints on the circle and some right triangle trig to determine the angles.




The integral is then,

$$\int_C \vec{F}\bullet d\vec{r} = \iint_D 6x^2 - \left(2xy - 6y^2\right)dA$$

$$= \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \int_0^3 6r^3 - 2r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$= \tfrac{1}{2}\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} 243 - 81\cos\theta\sin\theta\,d\theta = \boxed{\tfrac{81}{8}(6\pi-1)}$$

**3.**

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y^3z^4 & x\ln(z) & 10z-9y-8x \end{vmatrix}$$

$$= -9\,\vec{i} + 4x^2y^3z^3\,\vec{j} + \ln(z)\vec{k} - 3x^2y^2z^4\vec{k} + 8\vec{j} - \tfrac{x}{z}\vec{i}$$

$$= \boxed{-\left(9+\tfrac{x}{z}\right)\vec{i} + \left(4x^2y^3z^3+8\right)\vec{j} + \left(\ln(z) - 3x^2y^2z^4\right)\vec{k}}$$

$$\operatorname{div}\vec{F} = 2xy^3z^4 + 0 + 10 = \boxed{2xy^3z^4+10}$$

**4.** From our work on **#3** we can see that $\operatorname{curl}\vec{F} \ne \vec{0}$ and so the vector field is **not conservative**.

**5.** In this case the vector field **is conservative** because $\operatorname{curl}\vec{F} = \vec{0}$ as shown below.

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x\ln\left(y^2 z\right) & \dfrac{2x^2}{y} - 27y^2z^4 & \dfrac{x^2}{z} - 36y^3z^3 \end{vmatrix}$$

$$= -108y^2z^3\,\vec{i} + \tfrac{2x}{z}\,\vec{j} + \tfrac{4x}{y}\vec{k} - \tfrac{4x}{y}\vec{k} - \tfrac{2x}{z}\,\vec{j} + 108y^2z^3\vec{i} = \boxed{\vec{0}}$$

**6.** Let's call the points,

$$P = \left(-1,2,4\right) \qquad Q = \left(-1,0,3\right) \qquad R = \left(5,2,3\right)$$

The following two vectors will live in the plane.

$$\overrightarrow{PQ} = \left\langle 0,-2,-1\right\rangle \qquad \overrightarrow{PR} = \left\langle 6,0,-1\right\rangle$$

The cross product of these two will be normal to the plane.

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -2 & -1 \\ 6 & 0 & -1 \end{vmatrix} = 2\vec{i} - 6\vec{j} + 12\vec{k}$$

Using the point *Q* the equation of the plane is,

$$2\left(x+1\right) - 6y + 12\left(z-3\right) = 0 \quad\Rightarrow\quad 2x - 6y + 12z = 34 \quad\Rightarrow\quad z = \tfrac{1}{12}\left(34 - 2x + 6y\right)$$

The parameterization of the plane can then be written as,

$$\vec{r}\left(x,y\right) = \left\langle x, y, \tfrac{1}{12}\left(34 - 2x + 6y\right)\right\rangle$$

There will be no restrictions on *x* and *y* since we did not have any restrictions on the plane. Note as well that we could just have easily solved the equation of the plane for *x* or *y* and gotten two different (and potentially easier to deal with depending upon the problem) parameterizations.

**7.** The parameterization is : $\vec{r}\left(x,z\right) = \left\langle x, 7x^2 + 7z^2 - 9, z\right\rangle$. The limits on the variables come from the intersection.

$$10 = 7x^2 + 7z^2 - 9 \qquad\Rightarrow\qquad x^2 + z^2 = \tfrac{19}{7}$$

So, (*x*, *y*) come from the disk : $x^2 + z^2 \le \tfrac{19}{7}$.

**8.** Using "polar" conversion formulas the parameterization is :

$r\left(x,\theta\right) = \left\langle x, \sqrt{15}\sin\theta, \sqrt{15}\cos\theta\right\rangle$. The limits are : $0 \le \theta \le 2\pi,\ 20 \le x \le 30$.

**9.** Using the spherical conversion formulas the parameterization is :

$$\vec{r}\left(\theta,\varphi\right)=\left\langle 10\sin\varphi\cos\theta,10\sin\varphi\sin\theta,10\cos\varphi\right\rangle$$

The limits are : $\frac{3}{2}\pi\le\theta\le 2\pi,\ 0\le\varphi\le\pi$ . Note the limits on $\theta$ are because we must be in the 4th quadrant ( $x\ge 0$ and $y\le 0$ ). We also need the complete range of $\varphi$ because there were no restrictions on *z* mentioned and so we need to assume all possible values of *z*.

**#10.** First find $\vec{r}_u$ and $\vec{r}_v$ .

$$\vec{r}\left(u,v\right)=\left(u^2+4u-17\right)\vec{i}+\left(u^2+uv^3\right)\vec{j}+5v^2\vec{k}$$

$$\vec{r}_u\left(u,v\right)=\left(2u+4\right)\vec{i}+\left(2u+v^3\right)\vec{j}\qquad\qquad \vec{r}_v\left(u,v\right)=3uv^2\vec{j}+10v\vec{k}$$

Now, find the cross product of the two

$$\vec{r}_u\times\vec{r}_v=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2u+4 & 2u+v^3 & 0\\ 0 & 3uv^2 & 10v\end{vmatrix}=10v\left(2u+v^3\right)\vec{i}-10v\left(2u+4\right)\vec{j}+3uv^2\left(2u+4\right)\vec{k}$$

Now, find the values of *u* and *v* that give the point (15,128,20).

$$\begin{aligned} 15=u^2+4u-17 &\quad\Rightarrow\quad u=-8,\quad u=4\\ 128=u^2+uv^3 &\quad\Rightarrow\quad \underline{u=-8,\quad v=-2}\\ 20=5v^2 &\quad\Rightarrow\quad v=\pm 2\end{aligned}$$

Now, evaluate $\vec{r}_u\times\vec{r}_v$ at $u=-8$ and $v=-2$ : $\vec{r}_u\times\vec{r}_v=480\vec{i}-240\vec{j}+1152\vec{k}$

The equation of the tangent plane is then,

$$480\left(x-15\right)-240\left(y-128\right)+1152\left(z-20\right)=0\quad\rightarrow\quad 480x-240y+1153z=-480$$

**11.** First the parameterization : $\vec{r}\left(x,y\right)=\left\langle x,y,5-2x^2-2y^2\right\rangle$ and *D* is given by,

$$1=5-2x^2-2y^2\quad\Rightarrow\quad x^2+y^2=2\qquad\qquad 0\le\theta\le 2\pi,\quad 0\le r\le\sqrt{2}$$

We'll need polar eventually to do the integral based on *D*. Now all the cross product work…

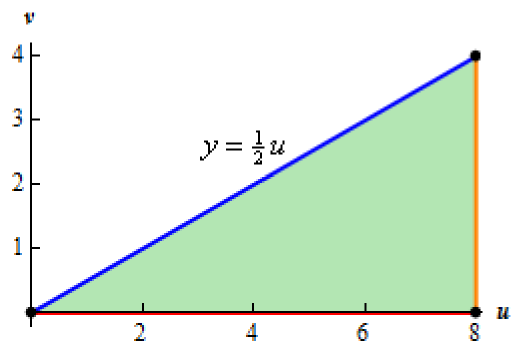
$$\vec{r}_x=\left\langle 1,0,-4x\right\rangle\qquad\qquad \vec{r}_y=\left\langle 0,1,-4y\right\rangle$$

$$\vec{r}_x\times\vec{r}_y=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & -4x\\ 0 & 1 & -4y\end{vmatrix}=4x\vec{i}+4y\vec{j}+\vec{k}\qquad\qquad \left\|\vec{r}_x\times\vec{r}_y\right\|=\sqrt{1+16x^2+16y^2}$$

The surface area is then,

$$S = \iint\limits_D \sqrt{1+16x^2+16y^2}\, dA = \int_0^{2\pi}\int_0^{\sqrt{2}} r\sqrt{1+16r^2}\, dr\, d\theta = \int_0^{2\pi} \tfrac{1}{48}\left(33^{\frac{3}{2}}-1\right) d\theta = \boxed{\tfrac{\pi}{24}\left(33^{\frac{3}{2}}-1\right)}$$

**12.** A sketch of the region $D$ is to the right. We already have a parameterization so the cross product work is,




$$\vec{r}_u = \langle 2u, 3, 0\rangle \qquad \vec{r}_v = \langle 0, -1, 2\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 3 & 0 \\ 0 & -1 & 2 \end{vmatrix} = 6\vec{i} - 4u\,\vec{j} - 2u\vec{k}$$

$$\left\|\vec{r}_u \times \vec{r}_v\right\| = \sqrt{36+16u^2+4u^2} = \sqrt{36+20u^2}$$

It looks like we'll need the following limits for the integral : $0 \le u \le 8, \quad 0 \le v \le \frac{1}{2}v$. The surface area is,

$$S = \iint\limits_D \sqrt{36+20u^2}\, dA = \int_0^8 \int_0^{\frac{1}{2}u} \sqrt{36+20u^2}\, dv\, du = \int_0^8 \tfrac{1}{2}u\sqrt{36+20u^2}\, du$$

$$= \boxed{\tfrac{1}{120}\left(1316^{\frac{3}{2}} - 216\right) = 396.035}$$