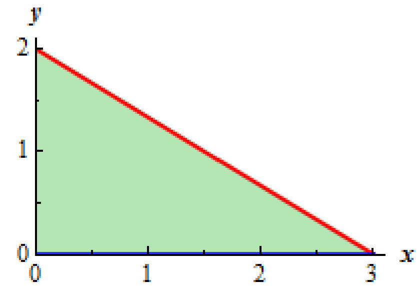
1. (3 pts) First, $z = 12 - 4x - 6y$. The integral is then,

$$\iint_S 8z - 2x\,dS = \iint_D \left(8\left(12-4x-6y\right)-2x\right)\sqrt{\left(-4\right)^2+\left(-6\right)^2+1}\,dA$$
$$= \sqrt{53}\iint_D 96 - 34x - 48y\,dA$$



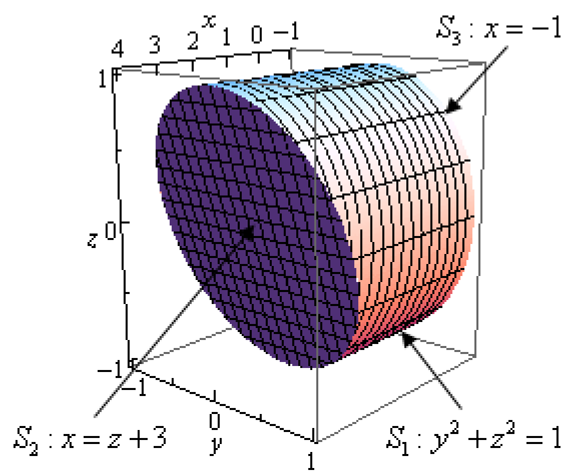
The region *D* is to the right and the limits are,

$$0 \le x \le 3 \qquad\qquad 0 \le y \le -\tfrac{2}{3}x + 2$$

Computing the integral gives,

$$\iint_S 8z - 2x\,dS = \sqrt{53}\iint_D 96 - 34x - 48y\,dA = \sqrt{53}\int_0^3\int_0^{-\frac{2}{3}x+2} 96 - 34x - 48y\,dy\,dx$$
$$= \sqrt{53}\int_0^3 192 - 132x + \tfrac{68}{3}x^2 - 24\left(2 - \tfrac{2}{3}x\right)^2 dx = \boxed{90\sqrt{53}}$$

3. (4 pts) A sketch of *S* is to the right. Here are the parameterizations for each portion of the surface.



S_1 : Cylinder

$$\vec{r}\left(x,\theta\right) = x\vec{i} + \sin\theta\,\vec{j} + \cos\theta\vec{k}$$
$$0 \le \theta \le 2\pi, -1 \le x \le z + 3 = \cos\theta + 3$$

S_2 : Front of cylinder

$x = 3 + z, \left(y,z\right)$ is in the disk of radius 1 centered at origin (in polar of course)

S_3 : Back of cylinder

$x = -1, \left(y,z\right)$ is in the disk of radius 1 centered at origin

Now, do the integral for each of these surfaces.

S_1 : Cylinder

In this case we'll need to do the cross product stuff so let's get that taken care of first.

$$\vec{r}_x = \vec{i} \qquad\qquad \vec{r}_\theta = \cos\theta\,\vec{j} - \sin\theta\vec{k}$$

$$\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \end{vmatrix} = \cos\theta\vec{k} + \sin\theta\vec{j}$$

$$\|\vec{r}_x \times \vec{r}_\theta\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\iint\limits_{S_1} x-3\,dS = \iint\limits_{D} (x-3)(1)\,dA = \int_0^{2\pi}\int_{-1}^{3+\cos\theta} x-3\,dx\,d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}(x-3)^2\Big|_{-1}^{3+\cos\theta} d\theta = \int_0^{2\pi} \tfrac{1}{2}\cos^2\theta - 8\,d\theta = \underline{-\tfrac{31\pi}{2}}$$

S_2 : Front of cylinder

$$\iint\limits_{S_2} x-3\,dS = \iint\limits_{D} (z)\sqrt{1+0+1}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^1 r^2\cos\theta\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\tfrac{1}{3}\cos(\theta)\,d\theta = \underline{0}$$

S_3 : Back of cylinder

$$\iint\limits_{S_3} x-3\,dS = \iint\limits_{D} (-1-3)\sqrt{1+0+0}\,dA = -4\iint\limits_{D} dA = \underline{-4\pi}$$

The integral is here is just the area of the disk of radius 1…..

So all together the integral is then,

$$\iint\limits_{S} x\,dS = -\tfrac{31\pi}{2} + 0 - 4\pi = \boxed{-\tfrac{39}{2}\pi}$$

4. (3 pts) First get the gradient,

$$f(x,y,z) = y - 2x^2 - 2z^2 + 9 \qquad\qquad \nabla f = \langle -4x, 1, -4x\rangle$$

Notice that this is oriented in the positive *y* direction (because the *y* component is positive) so we'll need to use the negative of this.

$$-\nabla f = \langle 4x, -1, 4z\rangle$$

The region *D* comes from

$$-1 = 2x^2 + 2z^2 - 9 \qquad\qquad \Rightarrow \qquad\qquad x^2 + z^2 = 4$$

So, *D* is a circle of radius 2 centered at the origin. Now the dot product.

$$\vec{F}\bullet\nabla f = \langle z, y^2, -x\rangle\bullet\langle 4x, -1, 4z\rangle = 4xz - y^2 - 4xy = -y^2 = -\left(2x^2 + 2z^2 - 9\right)^2$$

Notice that I didn't bother with the $\|\nabla f\|$ since they were just going to cancel when we go to do the integral. Speaking of which,

$$\iint\limits_{S} \vec{F}\cdot d\vec{S} = \iint\limits_{D} -\left(2x^2+2z^2-9\right)^2 dA = \int_0^{2\pi}\int_0^2 -r\left(2r^2-9\right)^2 dr\,d\theta = \int_0^{2\pi} -\tfrac{182}{3}\,d\theta = \boxed{-\tfrac{364\pi}{3}}$$

Not Graded

2. The integral is,

$$\iint\limits_{S} 2x^2 + 2z^2 - y\,dS = \iint\limits_{D} 8\sqrt{16x^2 + 1 + 16z^2}\,dA$$

where *D* is

$$0 = 2x^2 + 2z^2 - 8 \qquad \Rightarrow \qquad x^2 + z^2 = 4$$

So, *D* is the disk of radius 2 centered on the origin. Converting to polar coordinates then gives,

$$\iint\limits_S 2x^2 + 2z^2 - y\,dS = \iint\limits_D 8\sqrt{16x^2 + 1 + 16z^2}\,dA = \int_0^{2\pi}\int_0^2 8r\sqrt{1+16r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{6}\left(65^{\frac{3}{2}} - 1\right)d\theta = \boxed{\tfrac{\pi}{3}\left(65^{\frac{3}{2}} - 1\right)}$$

5. So, since we're using the surface from **#3** let's get that info copied to here.

S_1 : Cylinder

$$\vec{r}\left(x,\theta\right) = x\,\vec{i} + \sin\theta\,\vec{j} + \cos\theta\vec{k} \quad 0 \le \theta \le 2\pi, -1 \le x \le 3 + z = 3 + \cos\theta$$

S_2 : Front of cylinder

$$x = 3 + z, \quad D \text{ is the disk of radius 1 centered at origin (in polar of course)}$$

S_3 : Back of cylinder

$$x = -1, \quad D \text{ is the disk of radius 1 centered at origin}$$

Now, go through each of the integrals.

S_1 : Cylinder

In this case we'll need $\vec{r}_x \times \vec{r}_\theta$ which from **#3** is

$$\vec{r}_x \times \vec{r}_\theta = \sin\theta\,\vec{j} + \cos\theta\vec{k}$$

Note, that in this case this will always point outwards so we have the correct orientation. So the integral in this case becomes

$$\iint\limits_{S_1} \vec{F}\cdot d\vec{S} = \iint\limits_D \left\langle x, 0, -\cos\theta\right\rangle \cdot \left\langle 0, \sin\theta, \cos\theta\right\rangle dA = \iint\limits_D -\cos^2\theta dA$$

$$= \int_0^{2\pi}\int_{-1}^{3+\cos\theta} -\cos^2\theta\,dx\,d\theta = \int_0^{2\pi} -4\cos^2\left(\theta\right) - \cos^3\left(\theta\right)d\theta = \underline{-4\pi}$$

S_2 : Front of cylinder

The gradient is

$$f\left(x,y,z\right) = x - z - 3 \qquad \nabla f = \left\langle 1, 0, -1\right\rangle$$

This points in the positive *x* direction and so is pointing outward. The integral is now.

$$\iint\limits_{S_2} \vec{F}\cdot d\vec{S} = \iint\limits_D \left\langle 3+z, 0, -z\right\rangle \cdot \left\langle 1, 0, -1\right\rangle dA = \iint\limits_D 3 + 2z\,dA$$

$$= \int_0^{2\pi}\int_0^1 r\left(3 + 2r\cos\theta\right)dr\,d\theta = \int_0^{2\pi} \tfrac{2}{3}\cos\theta + \tfrac{3}{2}\,d\theta = \underline{3\pi}$$

S_3 : Back of cylinder

The gradient is

$$f\left(x,y,z\right) = x + 1 \qquad \nabla f = \left\langle 1, 0, 0\right\rangle \qquad -\nabla f = \left\langle -1, 0, 0\right\rangle$$

Since this is the back of the cylinder we need the orientation to be in the negative *x* direction to be outward. Note that we could just have easily done this directly using $\vec{n} = -\vec{i}$. The integral is

$$\iint\limits_{S_3} \vec{F}\cdot d\vec{S} = \iint\limits_D \left\langle -1, 0, -z\right\rangle \cdot \left\langle -1, 0, 0\right\rangle dA = \iint\limits_D dA = \underline{\pi}$$

As with **#3** note that the integral is the area of a disk of radius 1

So, the overall integral is

$$\iint\limits_S \vec{F}\cdot d\vec{S} = -4\pi + 3\pi + \pi = \boxed{0}$$

6. So, in this case we'll use Stokes' Theorem in the following direction.

$$\iint\limits_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int\limits_C \vec{F}\cdot d\vec{r}$$

where *C* is the boundary of the surface *S*. So, this will be the circle of radius $\sqrt{8}$ that lies at

$$x^2 + y^2 + z^2 = 20 \qquad \Rightarrow \qquad 8 + z^2 = 20 \qquad \Rightarrow \qquad z = \pm\sqrt{12}$$

Note that we only need the positive since we are told that it is the upper half of the sphere. The parameterization of the curve is

$$\vec{r}(t) = \left\langle \sqrt{8}\cos\theta, \sqrt{8}\sin\theta, \sqrt{12}\right\rangle \qquad \vec{r}'(t) = \left\langle -\sqrt{8}\sin\theta, \sqrt{8}\cos\theta, 0\right\rangle$$

Now, do the dot product is,

$$\begin{aligned}\vec{F}\cdot\vec{r}' &= \left\langle y^2, -y, 2x - 8z\right\rangle \cdot \left\langle -\sqrt{8}\sin\theta, \sqrt{8}\cos\theta, 0\right\rangle \\ &= -\sqrt{8}\sin\theta\left(y^2\right) + \sqrt{8}\cos\theta\left(-y\right) = -8\sqrt{8}\sin^3\theta - 8\cos\theta\sin\theta\end{aligned}$$

The integral is then

$$\iint\limits_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int\limits_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} -8\sqrt{8}\sin^3\theta - 8\cos\theta\sin\theta\, d\theta = \boxed{0}$$

7. This time we'll use Stokes' Theorem in the following direction

$$\int\limits_C \vec{F}\cdot d\vec{r} = \iint\limits_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

So, in this case we need the curl of the vector field.

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -2x & z^2 \end{vmatrix} = -2\vec{k} - \vec{k} = -3\vec{k}$$

Now, there are a variety of surfaces we could use here but it seems like one of the easiest to use is $y = 5 - x^2 + z^2$ and we'll need it to be oriented in the positive *y* direction (remember that the as we walk along the curve the surface must be on the left and our head will then point in the direction of the normal vector).

The gradient is then,

$$f(x,y,z) = y + x^2 + z^2 - 5 \qquad \nabla f = \left\langle 2x, 1, 2z\right\rangle$$

This gives the correct orientation and the region D is $x^2 + y^2 = 4$. So, the disk of radius 2 centered at the origin. The integral is then,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_D \langle 0,0,-3\rangle\cdot\langle 2x,1,2z\rangle\, dA = \iint_D -6z\, dA$$
$$= \int_0^{2\pi}\int_0^2 -6r^2\sin\theta\, dr\, d\theta = \int_0^{2\pi} -16\sin(\theta)\, d\theta = \boxed{0}$$

Note that we used the polar conversions : $x = r\cos\theta$, $z = r\sin\theta$.

8. We will be using the Divergence Theorem in the following direction.

$$\iint_S \vec{F}\cdot d\vec{r} = \iiint_E \operatorname{div}\vec{F} dV$$

So we will need the divergence and *E*.

$$\operatorname{div}\vec{F} = 3z^2 + 8x - 3z^2 = 8x$$

E is the portion of a sphere so we'll be doing this integral in spherical coordinates and the limits are,

$$0\le\varphi\le\frac{\pi}{2} \qquad 0\le\theta\le\frac{\pi}{2} \qquad 0\le\rho\le 3$$

The integral is then.

$$\iint_S \vec{F}\cdot d\vec{r} = \iiint_E 8x\, dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^3 8\rho^3\sin^2\varphi\cos\theta\, d\rho\, d\theta\, d\varphi$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} 162\sin^2\varphi\cos\theta\, d\theta\, d\varphi = \int_0^{\frac{\pi}{2}} 162\sin^2\varphi\, d\varphi = \boxed{\tfrac{81\pi}{2}}$$