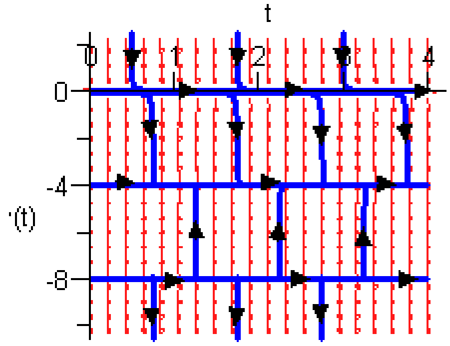
Math 3301 Homework Set 1 – Solutions 10 Points

1. (2 pts) Not much to this one. The derivative is zero at $y=-8$, $y=-4$, and $y=0$. Here's a sketch of the direction field as well as a few solutions. In class we often exaggerated the curvature but the program that I used didn't and so in some cases the curvature is not very apparent. Here is the long term behavior.



$y(0)\geq 0$	$y(t)\to 0$
$0<y(0)<-8$	$y(t)\to -4$
$y(0)=-8$	$y(t)=-8$
$y(0)<-8$	$y(t)\to -\infty$

4. (3 pts)

$$y'+\left(6-\tfrac{2}{t}\right)y=t^{3}\mathbf{e}^{-2t} \qquad \mu(t)=\mathbf{e}^{\int 6-\frac{2}{t}dt}=\mathbf{e}^{6t-2\ln|t|}=\mathbf{e}^{6t}\mathbf{e}^{-2\ln|t|}=t^{-2}\mathbf{e}^{6t}$$

$$\int\left(t^{-2}\mathbf{e}^{6t}y\right)'dt=\int t\mathbf{e}^{4t}\,dt \;\to\; t^{-2}\mathbf{e}^{6t}y=\mathbf{e}^{4t}\left(\tfrac{1}{4}t-\tfrac{1}{16}\right)+c \;\to\; y(t)=t^{2}\mathbf{e}^{-2t}\left(\tfrac{1}{4}t-\tfrac{1}{16}\right)+ct^{2}\mathbf{e}^{-6t}$$

$$2=y(1)=\mathbf{e}^{-2}\left(\tfrac{3}{16}\right)+c\mathbf{e}^{-6} \;\to\; c=2\mathbf{e}^{6}-\tfrac{3}{16}\mathbf{e}^{4} \;\Rightarrow\; \boxed{y(t)=t^{2}\mathbf{e}^{-2t}\left(\tfrac{1}{4}t-\tfrac{1}{16}\right)+\left(2\mathbf{e}^{6}-\tfrac{3}{16}\mathbf{e}^{4}\right)t^{2}\mathbf{e}^{-6t}}$$

5. (2 pts) We know from Calc I that relative extrema occur at critical points and critical points are those points where the derivative is zero or doesn't exist. However, we've been told that the derivative exists and is continuous everywhere so that means that at the critical point that gives the relative maximum, let's call it t_c, we must have $y'(t_c)=0$ and we want to know that $y(t_c)=40$ so all we need to do is "plug" t_c into the differential equation use what we know about the derivative and solve for t_c.

$$y'(t_c)-6y(t_c)=4-\mathbf{e}^{8t_c} \;\to\; \mathbf{e}^{8t_c}=244 \;\to\; \boxed{t_c=\tfrac{1}{8}\ln(244)=0.68715}$$

7. (3 pts)

$$\mu(t)=\mathbf{e}^{-6t} \qquad \int\left(\mathbf{e}^{-6t}y\right)'dt=\int 5\mathbf{e}^{-2t}dt \;\to\; \mathbf{e}^{-6t}y=-\tfrac{5}{2}\mathbf{e}^{-2t}+c$$

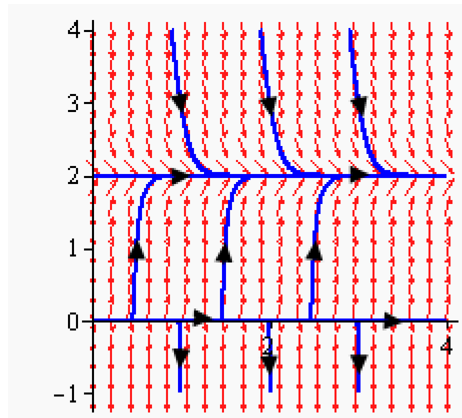
$$y(t)=-\tfrac{5}{2}\mathbf{e}^{4t}+c\mathbf{e}^{6t} \;\Rightarrow\; \boxed{y(t)=-\tfrac{5}{2}\mathbf{e}^{4t}+\left(\alpha^{2}-\tfrac{35}{2}\right)\mathbf{e}^{6t}}$$

Now, the first term will always approach $-\infty$ as $t\to\infty$. The second term will be the one that determines the long term behavior of the solution. If the coefficient of the second term is negative then the second term will also approach $-\infty$ as $t\to\infty$ and will the whole solution. If the coefficient of the second term is zero then that term will not be in the solution and again the solution will approach $-\infty$ as $t\to\infty$. Finally, if the coefficient of the second term is positive the second term will go approach ∞ as $t\to\infty$ and because the exponent in the second term is larger than the exponent in the first term and so will dominate the solution and the solution will then approach ∞ as $t\to\infty$. In terms of α this is,

$$\boxed{\begin{array}{l} -\sqrt{\tfrac{35}{2}} \le \alpha \le \sqrt{\tfrac{35}{2}} \;:\; y(t) \to -\infty \\ \alpha < -\sqrt{\tfrac{35}{2}}, \quad \alpha > \sqrt{\tfrac{35}{2}} \;:\; y(t) \to \infty \end{array}}$$

Not Graded

2. The derivative will be zero at $y = 0$ and $y = 2$. A sketch of the direction field and some solutions is to the right. From this we can see the following long term behavior and dependence on $y(0)$.



$y(0) > 0$	$y(t) \to 2$
$y(0) = 0$	$y(t) = 0$
$y(0) < 0$	$y(t) \to -\infty$

3.

$$y' - \tfrac{1}{x+2} y = \tfrac{1}{\sqrt{x+2}} \qquad \mu(t) = \mathbf{e}^{-\int \frac{1}{x+2} dx} = \mathbf{e}^{-\ln|x+2|} = \tfrac{1}{x+2}$$

$$\int \left(\tfrac{1}{x+2} y\right)' dx = \int \frac{1}{(x+2)^{\frac{3}{2}}} dx \quad \to \quad \tfrac{1}{x+2} y = \tfrac{-2}{\sqrt{x+2}} + c \quad \to \quad y(x) = -2\sqrt{x+2} + c(x+2)$$

$$0 = y(7) = -2(3) + 9c \quad \to \quad c = \tfrac{2}{3} \quad \Rightarrow \quad \boxed{y(x) = \tfrac{2}{3}(x+2) - 2\sqrt{x+2}}$$

6.

$$\mu(t) = \mathbf{e}^{\frac{1}{2}t} \qquad \int \left(\mathbf{e}^{\frac{1}{2}t} y\right)' dt = \int 8t\mathbf{e}^{\frac{1}{2}t} dt \quad \to \quad \mathbf{e}^{\frac{1}{2}t} y = 8\mathbf{e}^{\frac{1}{2}t}(2t - 4) + c$$

$$y(t) = 16t - 32 + c\mathbf{e}^{-\frac{1}{2}t} \quad \Rightarrow \quad \boxed{y(t) = 16t - 32 + (y_0 + 32)\mathbf{e}^{-\frac{1}{2}t}}$$

Okay, we've got a relative minimum at $t_c = 0.6$ and it is clear to see that the derivative will be continuous and so we know that we must then have $y'(t_c) = y'(0.6) = 0$ and so if we plug $t_c = 0.6$ into the differential equation we can solve for $y(0.6)$. At this point all we need to do is plug $t = 0.6$ into the solution and solve for y_0.

$$y'(0.6) + \tfrac{1}{2} y(0.6) = 8(0.6) = 4.8 \quad \to \quad y(0.6) = 9.6$$

Now all we need to do is plug into the solution and we'll be done.

$$9.6 = y(0.6) = -22.4 + (y_0 + 32)\mathbf{e}^{-0.3} \qquad \Rightarrow \qquad \boxed{y_0 = 11.1955}$$