2. (2 pts)

$$W = \begin{vmatrix} x & g \\ 1 & g' \end{vmatrix} = xg' - g = x^2\mathbf{e}^x \qquad \rightarrow \qquad g' - \tfrac{1}{x}g = x\mathbf{e}^x$$

This is a simple linear differential equation so I'll let you verify that the solution is,

$$\boxed{g(x) = x\mathbf{e}^x + cx}$$

4. (2 pts) I'll leave it to you to verify that $y_c(t) = c_1\mathbf{e}^{-3t} + c_2 t\mathbf{e}^{-3t}$. The guess for the particular solution and its derivatives are,

$$Y_P = A\mathbf{e}^{10t} \qquad Y_P' = 10A\mathbf{e}^{10t} \qquad Y_P'' = 100A\mathbf{e}^{10t}$$

Plugging this into the differential equation and simplifying gives,

$$169A\mathbf{e}^{10t} = -2\mathbf{e}^{10t}$$

Setting coefficients equal and solving gives,

$$169A = -2 \qquad \Rightarrow \qquad A = -\tfrac{2}{169}$$

The general solution is then,

$$\boxed{y(t) = c_1\mathbf{e}^{-3t} + c_2 t\mathbf{e}^{-3t} - \tfrac{2}{169}\mathbf{e}^{10t}}$$

6. (3 pts) I'll leave it to you to verify that $y_c(t) = c_1\mathbf{e}^{-\frac{1}{3}t} + c_2\mathbf{e}^{\frac{1}{3}t}$. The guess for the particular solution and its derivatives are,

$$Y_P = (At + B)\cos t + (Dt + E)\sin t \qquad Y_P' = (A + E + Dt)\cos t + (-B + D - At)\sin t$$

$$Y_P'' = (-B + 2D - At)\cos t + (-2A - E - Dt)\sin t$$

Plugging this into the differential equation and simplifying gives,

$$(-10B + 18D)\cos(t) + (-18A - 10E)\sin(t) - 10At\cos(t) - 10Dt\sin(t) = 4t\cos(t)$$

Setting coefficients equal and solving gives,

$$\begin{aligned} t\cos(t):& \quad -10A = 4 \\ t\sin(t):& \quad -10D = 0 \\ \cos(t):& \quad -10B + 18D = 0 \\ \sin(t):& \quad -18A - 10E = 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= -\tfrac{2}{5} \\ B &= 0 \\ D &= 0 \\ E &= \tfrac{18}{25} \end{aligned}$$

The general solution is then,

$$\boxed{y(t) = c_1\mathbf{e}^{-\frac{1}{3}t} + c_2\mathbf{e}^{\frac{1}{3}t} - \tfrac{2}{5}t\cos(t) + \tfrac{18}{25}\sin(t)}$$

7. (3 pts) I'll leave it to you to verify that $y_c(t) = c_1\mathbf{e}^{-5t} + c_2\mathbf{e}^{-t}$. The guess for the particular solution and its derivatives are,

$$Y_P = (At + B)\mathbf{e}^{3t} \qquad Y_P' = (A + 3B + 3At)\mathbf{e}^{3t} \qquad Y_P'' = (6A + 9B + 9At)\mathbf{e}^{3t}$$

Plugging this into the differential equation and simplifying gives,

$$(12A + 32B)\mathbf{e}^{3t} + 32At\mathbf{e}^{3t} = (1 - 8t)\mathbf{e}^{3t}$$

Setting coefficients equal and solving gives,

$$\begin{aligned} \mathbf{e}^{3t}: &\quad 12A+32B=1 \\ t\mathbf{e}^{3t}: &\quad 32A=-8 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A&=-\tfrac{1}{4} \\ B&=\tfrac{1}{8} \end{aligned}$$

The general solution is,

$$\underline{y(t)=c_1\mathbf{e}^{-5t}+c_2\mathbf{e}^{-t}+\left(\tfrac{1}{8}-\tfrac{1}{4}t\right)\mathbf{e}^{3t}}$$

Now apply the initial conditions.

$$\begin{aligned} 1&=y(0)=c_1+c_2+\tfrac{1}{8} \\ -6&=y'(0)=-5c_1-c_2+\tfrac{1}{8} \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1&=\tfrac{21}{16} \\ c_2&=-\tfrac{7}{16} \end{aligned}$$

The actual solution is then,

$$\boxed{y(t)=\tfrac{21}{16}\mathbf{e}^{-5t}-\tfrac{7}{16}\mathbf{e}^{-t}+\left(\tfrac{1}{8}-\tfrac{1}{4}t\right)\mathbf{e}^{3t}}$$

Not Graded

1. Compute the Wronskian.

$$W(y_1,y_2)=\begin{vmatrix} \mathbf{e}^{r_1 t} & \mathbf{e}^{r_2 t} \\ r_1\mathbf{e}^{r_1 t} & r_2\mathbf{e}^{r_2 t} \end{vmatrix}=r_2\mathbf{e}^{(r_1+r_2)t}-r_1\mathbf{e}^{(r_1+r_2)t}=(r_2-r_1)\mathbf{e}^{(r_1+r_2)t}\neq 0 \quad \text{b/c } r_2\neq r_1$$

So, they are a fundamental set of solutions and the general solution is what I claimed it to be and the general solution is $y(t)=c_1\mathbf{e}^{r_1 t}+c_2\mathbf{e}^{r_2 t}$.

3. I'll leave it to you to verify that $y_c(t)=c_1\mathbf{e}^{\frac{1}{3}t}\cos(t)+c_2\mathbf{e}^{\frac{1}{3}t}\sin(t)$. The guess for the particular and its derivatives are,

$$Y_P=At^2+Bt+C \qquad Y_P'=2At+B \qquad Y_P''=2A$$

Plugging this into the differential equation and simplifying gives,

$$10At^2+(-12A+10B)t+18A-6B+10C=20t^2+6$$

Setting coefficients equal and solving gives,

$$\begin{aligned} t^2: &\quad 10A=20 \\ t^1: &\quad -12A+10B=0 \\ t^0: &\quad 18A-6B+10C=6 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A&=2 \\ B&=\tfrac{12}{5} \\ C&=-\tfrac{39}{25} \end{aligned}$$

The general solution is then,

$$\boxed{y(t)=c_1\mathbf{e}^{\frac{1}{3}t}\cos(t)+c_2\mathbf{e}^{\frac{1}{3}t}\sin(t)+2t^2+\tfrac{12}{5}t-\tfrac{39}{25}}$$

5. I'll leave it to you to verify that $y_c(t)=c_1\mathbf{e}^{-6t}+c_2\mathbf{e}^{2t}$. The guess for the particular solution and its derivatives are,

$$Y_P=A\cos(4t)+B\sin(4t) \quad Y_P'=-4A\sin(4t)+4B\cos(4t) \quad Y_P''=-16A\cos(4t)-16B\sin(4t)$$

Plugging this into the differential equation and simplifying gives,

$$\left(-28A+16B\right)\cos\left(4t\right)+\left(-16A-28B\right)\sin\left(4t\right)=10\sin\left(4t\right)$$

Setting coefficients equal and solving gives,

$$\begin{aligned}\cos\left(4t\right):&\quad -28A+16B=0\\ \sin\left(4t\right):&\quad -16A-28B=10\end{aligned}\qquad\Rightarrow\qquad\begin{aligned}A&=-\tfrac{2}{13}\\ B&=-\tfrac{7}{26}\end{aligned}$$

The general solution is then,

$$\boxed{y\left(t\right)=c_1\mathbf{e}^{-6t}+c_2\mathbf{e}^{2t}-\tfrac{2}{13}\cos\left(4t\right)-\tfrac{7}{26}\sin\left(4t\right)}$$