2. (2 pts) I'll leave it to you to verify that $y_c(t) = c_1 + c_2\mathbf{e}^{7t}$. The guess for the particular solution and its derivatives are,

$$Y_P = A\mathbf{e}^{-2t} + Bt\mathbf{e}^{7t} \qquad Y_P' = -2A\mathbf{e}^{-2t} + B\mathbf{e}^{7t} + 7Bt\mathbf{e}^{7t} \qquad Y_P'' = 4A\mathbf{e}^{-2t} + 14B\mathbf{e}^{7t} + 49Bt\mathbf{e}^{7t}$$

Note that we need to tack a *t* onto the guess for the second exponential because it is part of the complimentary solution. Plugging this into the differential equation and simplifying gives,

$$18A\mathbf{e}^{-2t} + 7B\mathbf{e}^{7t} = 6\mathbf{e}^{-2t} - 3\mathbf{e}^{7t}$$

Setting coefficients equal and solving gives,

$$\begin{aligned} \mathbf{e}^{-2t}: &\quad 18A = 6 \\ \mathbf{e}^{7t}: &\quad 7B = -3 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= \tfrac{1}{3} \\ B &= -\tfrac{3}{7} \end{aligned}$$

The general solution is then,

$$\boxed{y(t) = c_1 + c_2\mathbf{e}^{7t} + \tfrac{1}{3}\mathbf{e}^{-2t} - \tfrac{3}{7}\mathbf{e}^{7t}}$$

5. (2 pts) I'll leave it to you to verify that $y_c(t) = c_1\cos(t) + c_2\sin(t)$. The guess for the particular solution is,

$$Y_P = (At + B)\mathbf{e}^{9t} + t\left(Ct^2 + Dt + E\right)\cos(t) + t\left(Ft^2 + Gt + H\right)\sin(t)$$

Note that we needed to tack a *t* onto the last two portions to prevent them from being in the complimentary solution.

7. (2 pts) I'll leave it to you to verify that $y_c(t) = c_1\cos\left(\tfrac{t}{3}\right) + c_2\sin\left(\tfrac{t}{3}\right)$. Here's the information we need for Variation of Parameters.

$$W = \begin{vmatrix} \cos\left(\tfrac{t}{3}\right) & \sin\left(\tfrac{t}{3}\right) \\ -\tfrac{1}{3}\sin\left(\tfrac{t}{3}\right) & \tfrac{1}{3}\cos\left(\tfrac{t}{3}\right) \end{vmatrix} = \tfrac{1}{3}\cos^2\left(\tfrac{t}{3}\right) + \tfrac{1}{3}\sin^2\left(\tfrac{t}{3}\right) = \tfrac{1}{3} \qquad g(t) = \tfrac{10}{9}$$

Don't forget to divide by the 9 to get a coefficient of 1 on the second derivative term. The particular solution is then,

$$\begin{aligned} Y_P &= -\cos\left(\tfrac{t}{3}\right)\int \frac{\tfrac{10}{9}\sin\left(\tfrac{t}{3}\right)}{\tfrac{1}{3}}\,dt + \sin\left(\tfrac{t}{3}\right)\int \frac{\tfrac{10}{9}\cos\left(\tfrac{t}{3}\right)}{\tfrac{1}{3}}\,dt \\ &= 10\cos^2\left(\tfrac{t}{3}\right) + 10\sin^2\left(\tfrac{t}{3}\right) = \underline{10} \end{aligned}$$

The general solution is,

$$\underline{y(t) = c_1\cos\left(\tfrac{t}{3}\right) + c_2\sin\left(\tfrac{t}{3}\right) + 10}$$

Now apply the initial conditions.

$$\begin{aligned} -8 &= y(0) = c_1 + 10 \\ 7 &= y'(0) = \tfrac{1}{3}c_2 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} c_1 &= -18 \\ c_2 &= 21 \end{aligned}$$

The actual solution is then,

$$\boxed{y(t) = -18\cos\left(\tfrac{t}{3}\right) + 21\sin\left(\tfrac{t}{3}\right) + 10}$$

8. (2 pts) I'll be leaving it to you to verify most of the solution work. The key quantities for the problem are,

$$m = \tfrac{18}{32} = \tfrac{9}{16} \qquad L = \tfrac{8}{12} = \tfrac{2}{3} \qquad k = \frac{18}{2/3} = 27 \qquad \omega_0 = \sqrt{\frac{27}{9/16}} = \sqrt{48} = 4\sqrt{3} = 6.9282$$

The IVP is,

$$\tfrac{9}{16}u'' + 27u = 0 \qquad u(0) = -\tfrac{2}{12} = -\tfrac{1}{6} \quad u'(0) = -\tfrac{8}{12} = -\tfrac{2}{3}$$

The general solution is,

$$u(t) = c_1 \cos\left(4\sqrt{3}\,t\right) + c_2 \sin\left(4\sqrt{3}\,t\right)$$

Applying the initial conditions gives,

$$\underline{u(t) = -\tfrac{1}{6}\cos\left(4\sqrt{3}\,t\right) - \tfrac{1}{6\sqrt{3}}\sin\left(4\sqrt{3}\,t\right)}$$

Now, reduce down to a single cosine.

$$R = \sqrt{\left(-\tfrac{1}{6}\right)^2 + \left(-\tfrac{1}{6\sqrt{3}}\right)^2} = \tfrac{1}{3\sqrt{3}} = 0.1925 \qquad \delta_1 = \tan^{-1}\left(\frac{-\frac{1}{6\sqrt{3}}}{-\frac{1}{6}}\right) = 0.5236 \qquad \delta_2 = \delta_1 + \pi = 3.6652$$

In this case the second angle is the correct one and so the single cosine form of the solution is,

$$\boxed{u(t) = 0.1925\cos\left(4\sqrt{3}\,t - 3.6652\right)}$$

9. (2 pts) I'll be leaving it to you to verify most of the solution work. The key quantities for the problem are,

$$m = 0.8 \qquad L = 0.25 \qquad k = \frac{(0.8)(9.8)}{0.25} = 31.36 \qquad \gamma = \tfrac{6}{0.3} = 20 \qquad \gamma_{CR} = 2\sqrt{(31.36)(0.8)} = 10.0176$$

We have over damping in this case. The IVP is then,

$$0.8u'' + 20u' + 31.36u = 0 \qquad u(0) = 0 \quad u'(0) = 0.1$$

The general solution is,

$$u(t) = c_1\mathbf{e}^{-23.3190t} + c_2\mathbf{e}^{-1.6810t}$$

Applying the initial conditions gives,

$$\boxed{u(t) = -0.004621\mathbf{e}^{-23.3190t} + 0.004621\mathbf{e}^{-1.6810t}}$$

Not Graded

1. I'll leave it to you to verify that $y_c(t) = c_1 + c_2\mathbf{e}^{7t}$. The guess for the particular solution and its derivatives are,

$$Y_P = t(At + B) + D\cos(2t) + E\sin(2t) \qquad Y_P' = 2At + B - 2D\sin(2t) + 2E\cos(2t)$$
$$Y_P'' = 2A - 4D\cos(2t) - 4E\sin(2t)$$

Note that we need to tack a *t* onto the guess for the polynomial because the constant in the guess is part of the complimentary solution. Plugging this into the differential equation and simplifying gives,

$$-7A + 2B - 14Bt + (-4D - 14E)\cos(2t) + (14D - 4E)\sin(2t) = 5t - 1 + \cos(2t)$$

Setting coefficients equal and solving gives,

$$\begin{aligned} t^1: &\quad -14A = 5 \\ t^0: &\quad -7A + 2B = -1 \\ \cos(2t): &\quad -4D - 14E = 1 \\ \sin(2t): &\quad 14D - 4E = 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= -\tfrac{5}{14} \\ B &= \tfrac{2}{49} \\ D &= -\tfrac{1}{53} \\ E &= -\tfrac{7}{106} \end{aligned}$$

The general solution is then,

$$\boxed{y(t) = c_1 + c_2\mathbf{e}^{7t} - \tfrac{5}{14}t^2 + \tfrac{2}{49}t - \tfrac{1}{53}\cos(2t) - \tfrac{7}{106}\sin(2t)}$$

3. I'll leave it to you to verify that $y_c(t) = c_1\mathbf{e}^{2t} + c_2 t\mathbf{e}^{2t}$. The guess for the particular solution and its derivatives are,

$$Y_P = At^2\mathbf{e}^{2t} \qquad Y_P' = 2At\mathbf{e}^{2t} + 2At^2\mathbf{e}^{2t} \qquad Y_P'' = 2A\mathbf{e}^{2t} + 8At\mathbf{e}^{2t} + 4At^2\mathbf{e}^{2t}$$

Note that we needed to tack a t^2 onto the guess so we could get a guess that did not end up in the complimentary solution. Plugging this into the differential equation and simplifying gives,

$$2A\mathbf{e}^{2t} = 7\mathbf{e}^{2t}$$

Setting coefficients equal and solving gives,

$$2A = 7 \qquad \Rightarrow \qquad A = \tfrac{7}{2}$$

The general solution is,

$$\underline{y(t) = c_1\mathbf{e}^{2t} + c_2 t\mathbf{e}^{2t} + \tfrac{7}{2}t^2\mathbf{e}^{2t}}$$

Now apply the initial conditions.

$$\begin{aligned} 12 &= y(0) = c_1 \\ -1 &= y'(0) = 2c_1 + c_2 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} c_1 &= 12 \\ c_2 &= -25 \end{aligned}$$

The actual solution is then,

$$\boxed{y(t) = 12\mathbf{e}^{2t} - 25t\mathbf{e}^{2t} + \tfrac{7}{2}t^2\mathbf{e}^{2t}}$$

4. I'll leave it to you to verify that $y_c(t) = c_1\mathbf{e}^{4t}\cos(3t) + c_2\mathbf{e}^{4t}\sin(3t)$. The guess for the particular solution is,

$$Y_P = \left(At^2 + Bt + C\right)\mathbf{e}^{4t} + (Dt + E)\cos(3t) + (Ft + G)\sin(3t) + t\mathbf{e}^{4t}\left(H\cos(3t) + I\sin(3t)\right)$$

Note that we needed to tack a *t* onto the last guess to prevent that part from being in the complimentary solution.

6. I'll leave it to you to verify that $y_c(t) = c_1\mathbf{e}^{-2t} + c_2\mathbf{e}^{4t}$. Here's the information we need for Variation of Parameters.

$$W = \begin{vmatrix} \mathbf{e}^{-2t} & \mathbf{e}^{4t} \\ -2\mathbf{e}^{-2t} & 4\mathbf{e}^{4t} \end{vmatrix} = 6\mathbf{e}^{2t} \qquad g(t) = \mathbf{e}^{4t} - 3\mathbf{e}^{-t}$$

The particular solution is then,

$$Y_P = -\mathbf{e}^{-2t}\int \frac{\mathbf{e}^{4t}\left(\mathbf{e}^{4t}-3\mathbf{e}^{-t}\right)}{6\mathbf{e}^{2t}}\,dt + \mathbf{e}^{4t}\int \frac{\mathbf{e}^{-2t}\left(\mathbf{e}^{4t}-3\mathbf{e}^{-t}\right)}{6\mathbf{e}^{2t}}\,dt$$

$$= -\tfrac{1}{6}\mathbf{e}^{-2t}\int \mathbf{e}^{6t}-3\mathbf{e}^{t}\,dt + \tfrac{1}{6}\mathbf{e}^{4t}\int 1-3\mathbf{e}^{-5t}\,dt = \underline{\tfrac{1}{6}\left[t\mathbf{e}^{4t} - \tfrac{1}{6}\mathbf{e}^{4t} + \tfrac{18}{5}\mathbf{e}^{-t}\right]}$$

The general solution is then,

$$\boxed{y(t) = c_1\mathbf{e}^{-2t} + c_2\mathbf{e}^{4t} + \tfrac{1}{6}\left[t\mathbf{e}^{4t} - \tfrac{1}{6}\mathbf{e}^{4t} + \tfrac{18}{5}\mathbf{e}^{-t}\right]}$$

10. Taking what we can from **#8** we have the following,

$$\tfrac{9}{16}u'' + 27u = 0 \quad u(0) = -\tfrac{1}{6} \quad u'(0) = -\tfrac{2}{3} \qquad \underline{u_c(t) = c_1\cos\left(4\sqrt{3}\,t\right) + c_2\sin\left(4\sqrt{3}\,t\right)}$$

Note that because $\omega = 2 \neq 4\sqrt{3} = \omega_0$ we will **NOT** have resonance. Undetermined Coefficients will probably be the easiest for a particular solution so,

$$U_P = A\cos(2t) + B\sin(2t) \quad \to \quad \tfrac{99}{4}A\cos(2t) + \tfrac{99}{4}B\sin(2t) = 9\cos(2t) + 4\sin(2t)$$

Setting coefficients equal and solving gives $A = \tfrac{4}{11}$, $B = \tfrac{16}{99}$. The general solution is then,

$$u(t) = c_1\cos\left(4\sqrt{3}\,t\right) + c_2\sin\left(4\sqrt{3}\,t\right) + \tfrac{4}{11}\cos(2t) + \tfrac{16}{99}\sin(2t)$$

Applying the initial conditions gives,

$$u(t) = -\tfrac{35}{66}\cos\left(4\sqrt{3}\,t\right) - \tfrac{49}{198\sqrt{3}}\sin\left(4\sqrt{3}\,t\right) + \tfrac{4}{11}\cos(2t) + \tfrac{16}{99}\sin(2t)$$

Now, reduce the first set of sine/cosine down to a single cosine.

$$R = \sqrt{\left(-\tfrac{35}{66}\right)^2 + \left(-\tfrac{49}{198\sqrt{3}}\right)^2} = 5.9608 \qquad \delta_1 = \tan^{-1}\left(\frac{-49/198\sqrt{3}}{-35/66}\right) = 0.2632 \quad \delta_2 = \delta_1 + \pi = 3.4048$$

In this case the second angle is correct. Next reduce the second set of sine/cosine.

$$R = \sqrt{\left(\tfrac{4}{11}\right)^2 + \left(\tfrac{16}{99}\right)^2} = 0.3979 \qquad \delta_1 = \tan^{-1}\left(\frac{16/99}{4/11}\right) = 0.4182 \quad \delta_2 = \delta_1 + \pi = 3.5598$$

Here the first angle is correct. The solution is then,

$$\boxed{u(t) = 5.9608\cos\left(4\sqrt{3}\,t - 3.4048\right) + 0.3979\cos(2t - 0.4182)}$$

11. Taking what we can from **#8** we have the following,

$$0.8u'' + 31.36u = 0 \quad u(0) = 0 \quad u'(0) = 0.1 \qquad \underline{u_c(t) = c_1\mathbf{e}^{-23.3190t} + c_2\mathbf{e}^{-1.6810t}}$$

Undetermined Coefficients will probably be the easiest for a particular solution so,

$$U_P = A\cos(4t) + B\sin(4t) \to (18.56A + 80B)A\cos(4t) + (18.56B - 80A)B\sin(4t) = 11\sin(4t)$$

Setting coefficients equal and solving gives $A = -0.1305$, $B = 0.0303$. The general solution is then,

$$u(t) = c_1\mathbf{e}^{-23.3190t} + c_2\mathbf{e}^{-1.6810t} - 0.1305\cos(2t) + 0.0303\sin(2t)$$

Applying the initial conditions gives,

$$u(t) = -0.009162\mathbf{e}^{-23.3190t} + 0.1396\mathbf{e}^{-1.6810t} - 0.1305\cos(2t) + 0.0303\sin(2t)$$

Now reduce the sine/cosine down to a single cosine.

$$R = \sqrt{\left(-0.1305\right)^2 + \left(0.0303\right)^2} = 0.1340 \quad \delta_1 = \tan^{-1}\left(\tfrac{0.0303}{-0.1305}\right) = -0.2281 \quad \delta_2 = \delta_1 + \pi = 2.9135$$

In this case the second angle is correct. The solution is then,

$$\boxed{u\left(t\right) = -0.009162\mathbf{e}^{-23.3190t} + 0.1396\mathbf{e}^{-1.6810t} - 0.1340\cos\left(2t - 2.9135\right)}$$