

7. (3 pts) I'll leave it to you to verify that the eigenvalues and eigenvectors for this matrix are,

$$\lambda_1 = -9 \quad \vec{\eta}^{(1)} = \begin{pmatrix} -5 \\ 1 \end{pmatrix} \quad \lambda_2 = -2 \quad \vec{\eta}^{(2)} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

The general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-9t} \begin{pmatrix} -5 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{-2t} \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

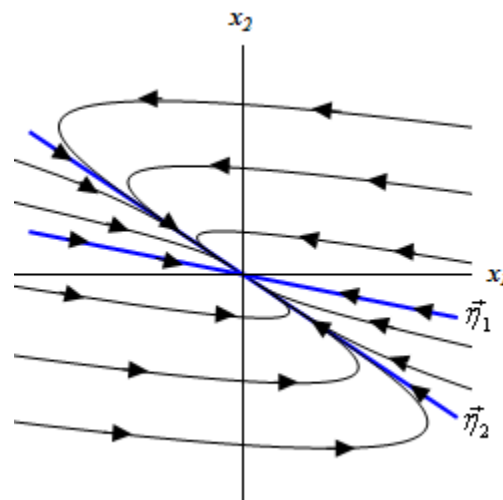
Applying the initial conditions gives,

$$\begin{pmatrix} 0 \\ -4 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} -5 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 2 \end{pmatrix} \Rightarrow \begin{aligned} 0 &= -5c_1 - 3c_2 \\ -4 &= c_1 + 2c_2 \end{aligned} \Rightarrow \begin{aligned} c_1 &= \frac{12}{7} \\ c_2 &= -\frac{20}{7} \end{aligned}$$

The actual solution is then,

$$\vec{x}(t) = \frac{12}{7} \mathbf{e}^{-9t} \begin{pmatrix} -5 \\ 1 \end{pmatrix} - \frac{20}{7} \mathbf{e}^{-2t} \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

A sketch of the phase portrait is to the right. The equilibrium solution in this case is an **asymptotically stable node**. Don't get excited about the fact that the two lines are so close together. Sometimes that happens.



10. (4 pts) First convert to a system in matrix form.

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \end{aligned} \Rightarrow \begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{17}{4}x_1 + x_2 \end{aligned} \Rightarrow \vec{x}' = \begin{bmatrix} 0 & 1 \\ -\frac{17}{4} & 1 \end{bmatrix} \vec{x} \quad \vec{x}(0) = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

I'll leave it to you to verify that the eigenvalues and eigenvectors for this matrix are,

$$\lambda_1 = \frac{1}{2} + 2i \quad \vec{\eta}^{(1)} = \begin{pmatrix} 2 \\ 1 + 4i \end{pmatrix} \quad \lambda_2 = \frac{1}{2} - 2i \quad \vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 1 - 4i \end{pmatrix}$$

Next,

$$\begin{aligned} \mathbf{e}^{(\frac{1}{2}+2i)t} \begin{pmatrix} 2 \\ 1+4i \end{pmatrix} &= \mathbf{e}^{\frac{1}{2}t} (\cos(2t) + i \sin(2t)) \begin{pmatrix} 2 \\ 1+4i \end{pmatrix} \\ &= \mathbf{e}^{\frac{1}{2}t} \begin{pmatrix} 2 \cos(2t) \\ \cos(2t) - 4 \sin(2t) \end{pmatrix} + i \mathbf{e}^{\frac{1}{2}t} \begin{pmatrix} 2 \sin(2t) \\ 4 \cos(2t) + \sin(2t) \end{pmatrix} \end{aligned}$$

The general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{\frac{1}{2}t} \begin{pmatrix} 2 \cos(2t) \\ \cos(2t) - 4 \sin(2t) \end{pmatrix} + c_2 \mathbf{e}^{\frac{1}{2}t} \begin{pmatrix} 2 \sin(2t) \\ 4 \cos(2t) + \sin(2t) \end{pmatrix}$$

Applying the initial conditions gives,

$$\begin{pmatrix} 5 \\ -1 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 4 \end{pmatrix} \Rightarrow \begin{aligned} 5 &= 2c_1 \\ -1 &= c_1 + 4c_2 \end{aligned} \Rightarrow \begin{aligned} c_1 &= \frac{5}{2} \\ c_2 &= -\frac{7}{8} \end{aligned}$$

The actual solution to the system is then,

$$\vec{x}(t) = \frac{5}{2} e^{\frac{1}{2}t} \begin{pmatrix} 2 \cos(2t) \\ \cos(2t) - 4 \sin(2t) \end{pmatrix} - \frac{7}{8} e^{\frac{1}{2}t} \begin{pmatrix} 2 \sin(2t) \\ 4 \cos(2t) + \sin(2t) \end{pmatrix} = \begin{pmatrix} 5 \cos(2t) - \frac{7}{4} \sin(2t) \\ -\cos(2t) - \frac{87}{8} \sin(2t) \end{pmatrix}$$

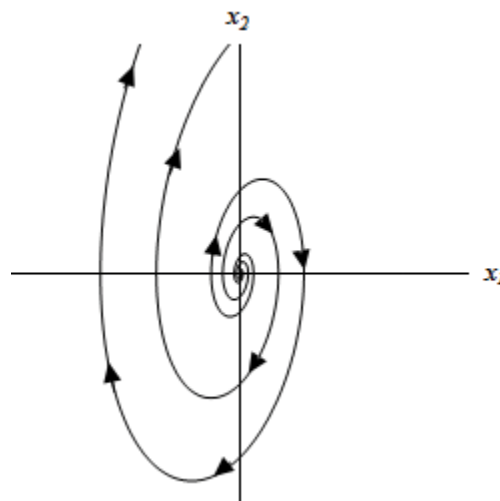
The solution to the original differential equation is then,

$$\boxed{\begin{aligned} y(t) &= e^{\frac{1}{2}t} \left(5 \cos(2t) - \frac{7}{4} \sin(2t) \right) \\ &= 5e^{\frac{1}{2}t} \cos(2t) - \frac{7}{4} e^{\frac{1}{2}t} \sin(2t) \end{aligned}}$$

From the system,

$$\begin{bmatrix} 0 & 1 \\ -\frac{17}{4} & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{17}{4} \end{pmatrix}$$

we can see that trajectories will cross the x_1 axis going downwards. A sketch of the phase portrait is to the right. The equilibrium solution in this case is an **unstable spiral**.



11. (3 pts) I'll leave it to you to verify that the eigenvalues and eigenvectors for this matrix are,

$$\lambda_{1,2} = 5 \quad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Next,

$$\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow 4\rho_1 + 4\rho_2 = 1 \Rightarrow \rho_1 = \frac{1}{4} - \rho_2 \Rightarrow \vec{\rho} = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$$

The general solution is then,

$$\vec{x}(t) = c_1 e^{5t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left\{ t e^{5t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{5t} \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} \right\}$$

Applying the initial conditions gives,

$$\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \vec{x}(0) = c_1 e^{5 \cdot 0} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} 0 &= -c_1 + \frac{1}{4}c_2 \\ -2 &= c_1 \end{aligned} \Rightarrow \begin{aligned} c_1 &= -2 \\ c_2 &= -8 \end{aligned}$$

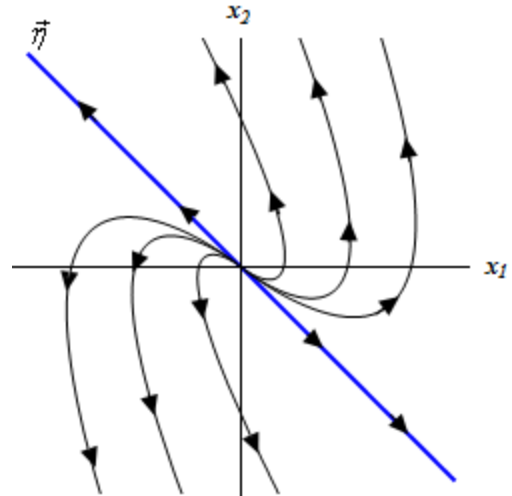
The actual solution is then,

$$\begin{aligned}\vec{x}(t) &= -2\mathbf{e}^{5t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} - 8 \left\{ t\mathbf{e}^{5t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mathbf{e}^{5t} \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} \right\} \\ &= \mathbf{e}^{-2t} \begin{pmatrix} 8t \\ -2-8t \end{pmatrix}\end{aligned}$$

From the system,

$$\begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

we can see that trajectories will cross the x_1 axis going upwards. A sketch of the phase portrait is to the right. The equilibrium solution in this case is an **unstable improper node**.



Not Graded

1. First the eigenvalues.

$$\det(A - \lambda I) = \begin{vmatrix} \frac{3}{2} - \lambda & 5 \\ \frac{1}{2} & 3 - \lambda \end{vmatrix} = \left(\frac{3}{2} - \lambda\right)(3 - \lambda) - \frac{5}{2} = \left(\lambda - \frac{1}{2}\right)(\lambda - 4) \Rightarrow \lambda_1 = \frac{1}{2}, \lambda_2 = 4$$

Now the eigenvector for $\lambda_1 = \frac{1}{2}$.

$$\begin{bmatrix} 1 & 5 \\ \frac{1}{2} & \frac{5}{2} \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \eta_1 + 5\eta_2 = 0 \Rightarrow \eta_1 = -5\eta_2 \Rightarrow \vec{\eta}^{(1)} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$$

the eigenvector for $\lambda_2 = 4$.

$$\begin{bmatrix} -\frac{5}{2} & 5 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \frac{1}{2}\eta_1 - \eta_2 = 0 \Rightarrow \eta_1 = 2\eta_2 \Rightarrow \vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

2. First the eigenvalues.

$$\det(A - \lambda I) = \begin{vmatrix} -2 - \lambda & 1 \\ -1 & -4 - \lambda \end{vmatrix} = (-2 - \lambda)(-4 - \lambda) + 1 = (\lambda + 3)^2 \Rightarrow \lambda_{1,2} = -3$$

The (only) eigenvector for $\lambda_{1,2} = -3$ is then,

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \eta_1 + \eta_2 = 0 \Rightarrow \eta_1 = -\eta_2 \Rightarrow \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

3. First the eigenvalues.

$$\det(A - \lambda I) = \begin{vmatrix} 5 - \lambda & -5 \\ 10 & -9 - \lambda \end{vmatrix} = (5 - \lambda)(-9 - \lambda) + 50 = \lambda^2 + 4\lambda + 5 \Rightarrow \lambda_{1,2} = 2 \pm i$$

Now the eigenvector for $\lambda_1 = 2 + i$

$$\begin{bmatrix} 3-i & -5 \\ 10 & -11-i \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (3-i)\eta_1 - 5\eta_2 = 0 \Rightarrow \eta_2 = \frac{1}{5}(3-i)\eta_1 \Rightarrow \vec{\eta}^{(1)} = \begin{pmatrix} 3-i \\ 5 \end{pmatrix}$$

Then by the fact given in class the eigenvector for $\lambda_2 = 2 - i$ is,

$$\vec{\eta}^{(2)} = \begin{pmatrix} 3+i \\ 5 \end{pmatrix}$$

4.

$$\begin{array}{lcl} x_1 = y & \Rightarrow & x'_1 = x_2 \\ x_2 = y' & \Rightarrow & x'_2 = 3x_1 + \frac{9}{2}x_2 \end{array} \Rightarrow \vec{x}' = \begin{bmatrix} 0 & 1 \\ 3 & \frac{9}{2} \end{bmatrix} \vec{x} \quad \vec{x}(0) = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$$

5.

$$\begin{array}{lcl} x_1 = y & \Rightarrow & x'_1 = x_2 \\ x_2 = y' & \Rightarrow & x'_2 = x_3 \\ x_3 = y'' & \Rightarrow & x'_3 = x_2 - 14x_3 \end{array} \Rightarrow \vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -14 \end{bmatrix} \vec{x} \quad \vec{x}(0) = \begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix}$$

6. I'll leave it to you to verify that the (messy) eigenvalues and eigenvectors for this matrix are,

$$\lambda_1 = \frac{1}{2}(11 + \sqrt{97}) \quad \vec{\eta}^{(1)} = \begin{pmatrix} 6 \\ 7 + \sqrt{97} \end{pmatrix} \quad \lambda_2 = \frac{1}{2}(11 - \sqrt{97}) \quad \vec{\eta}^{(2)} = \begin{pmatrix} 6 \\ 7 - \sqrt{97} \end{pmatrix}$$

The general solution is then,

$$\boxed{\vec{x}(t) = c_1 e^{\frac{1}{2}(11+\sqrt{97})t} \begin{pmatrix} 6 \\ 7 + \sqrt{97} \end{pmatrix} + c_2 e^{\frac{1}{2}(11-\sqrt{97})t} \begin{pmatrix} 6 \\ 7 - \sqrt{97} \end{pmatrix}}$$

Don't get excited when you run into "messy" eigenvalues and/or eigenvectors. They are fairly common.

8. I'll leave it to you to verify that the eigenvalues and eigenvectors for this matrix are,

$$\lambda_1 = -3 \quad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 7 \end{pmatrix} \quad \lambda_2 = 5 \quad \vec{\eta}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The general solution is then,

$$\vec{x}(t) = c_1 e^{-3t} \begin{pmatrix} -1 \\ 7 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

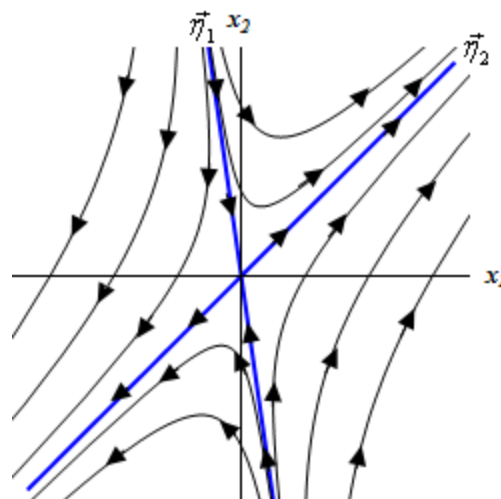
Applying the initial conditions gives,

$$\begin{pmatrix} -3 \\ -1 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} -1 \\ 7 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{l} -3 = -c_1 + c_2 \\ -1 = 7c_1 + c_2 \end{array} \Rightarrow \begin{array}{l} c_1 = \frac{1}{4} \\ c_2 = -\frac{11}{4} \end{array}$$

The actual solution is then,

$$\vec{x}(t) = \frac{1}{4}e^{-3t} \begin{pmatrix} -1 \\ 7 \end{pmatrix} - \frac{11}{4}e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

A sketch of the phase portrait is to the right. The equilibrium solution in this case is an **unstable saddle point**. Don't get excited about the fact that the two lines are so close together. Sometimes that happens.



9. I'll leave it to you to verify that the eigenvalues and eigenvectors for this matrix are,

$$\lambda_1 = 4i \quad \vec{\eta}^{(1)} = \begin{pmatrix} -1+2i \\ 2 \end{pmatrix} \quad \lambda_2 = -4i \quad \vec{\eta}^{(2)} = \begin{pmatrix} -1-2i \\ 2 \end{pmatrix}$$

Next,

$$\begin{aligned} e^{4it} \begin{pmatrix} -1+2i \\ 2 \end{pmatrix} &= (\cos(4t) + i \sin(4t)) \begin{pmatrix} -1+2i \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -\cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix} + i \begin{pmatrix} 2\cos(4t) - \sin(4t) \\ 2\sin(4t) \end{pmatrix} \end{aligned}$$

The general solution is then,

$$\vec{x}(t) = c_1 \begin{pmatrix} -\cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix} + c_2 \begin{pmatrix} 2\cos(4t) - \sin(4t) \\ 2\sin(4t) \end{pmatrix}$$

Applying the initial conditions gives,

$$\begin{pmatrix} 7 \\ -4 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} 7 &= -c_1 + 2c_2 \\ -4 &= 2c_1 \end{aligned} \Rightarrow \begin{aligned} c_1 &= -2 \\ c_2 &= \frac{5}{2} \end{aligned}$$

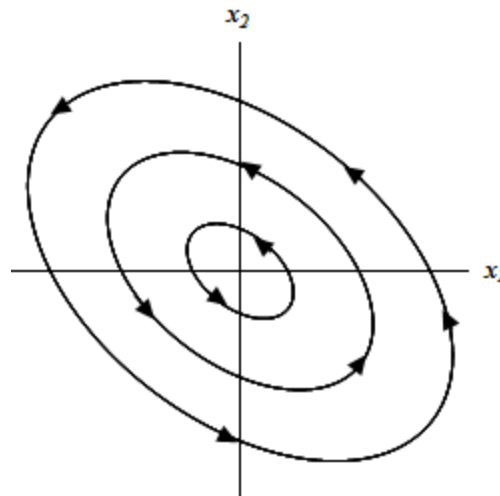
The actual solution is then,

$$\vec{x}(t) = -2 \begin{pmatrix} -\cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix} + \frac{5}{2} \begin{pmatrix} 2\cos(4t) - \sin(4t) \\ 2\sin(4t) \end{pmatrix} = \begin{pmatrix} 7\cos(4t) + \frac{3}{2}\sin(4t) \\ -4\cos(4t) + 5\sin(4t) \end{pmatrix}$$

From the system,

$$\begin{bmatrix} -2 & -5 \\ 4 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

we can see that trajectories will cross the x_1 axis going upwards. A sketch of the phase portrait is to the right. The equilibrium solution in this case is an **stable center**.



12. I'll leave it to you to verify that the eigenvalues and eigenvectors for this matrix are,

$$\lambda_{1,2} = -\frac{1}{2} \quad \vec{\eta}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Next,

$$\begin{bmatrix} -\frac{3}{4} & \frac{3}{2} \\ -\frac{3}{8} & \frac{3}{4} \end{bmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow -\frac{3}{4}\rho_1 + \frac{3}{2}\rho_2 = 2 \Rightarrow \rho_1 = -\frac{8}{3} - 2\rho_2 \Rightarrow \vec{\rho} = \begin{pmatrix} -\frac{8}{3} \\ 0 \end{pmatrix}$$

The general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-\frac{1}{2}t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left\{ t \mathbf{e}^{-\frac{1}{2}t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \mathbf{e}^{-\frac{1}{2}t} \begin{pmatrix} -\frac{8}{3} \\ 0 \end{pmatrix} \right\}$$

Applying the initial conditions gives,

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -\frac{8}{3} \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} 4 &= 2c_1 - \frac{8}{3}c_2 \\ 1 &= c_1 \end{aligned} \Rightarrow \begin{aligned} c_1 &= 1 \\ c_2 &= -\frac{3}{4} \end{aligned}$$

The actual solution is then,

$$\begin{aligned} \vec{x}(t) &= \mathbf{e}^{-\frac{1}{2}t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{3}{4} \left\{ t \mathbf{e}^{-\frac{1}{2}t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \mathbf{e}^{-\frac{1}{2}t} \begin{pmatrix} -\frac{8}{3} \\ 0 \end{pmatrix} \right\} \\ &= \mathbf{e}^{-\frac{1}{2}t} \begin{pmatrix} 4 - \frac{3}{2}t \\ 1 - \frac{3}{4}t \end{pmatrix} \end{aligned}$$

From the system,

$$\begin{bmatrix} -\frac{5}{4} & \frac{3}{2} \\ -\frac{3}{8} & \frac{1}{4} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{5}{4} \\ -\frac{3}{8} \end{pmatrix}$$

we can see that trajectories will cross the x_1 axis going downwards. A sketch of the phase portrait is to the right. The equilibrium solution in this case is an **asymptotically stable improper node**.

