1. (2 pts) $y = \frac{2}{\sqrt{3}}\tan\theta \qquad dy = \frac{2}{\sqrt{3}}\sec^2\theta \qquad \sqrt{4+3y^2} = \sqrt{4+4\tan^2\theta} = 2\left|\sec\theta\right| = 2\sec\theta$

$$\int \frac{\sqrt{4+3y^2}}{y^6}\,dy = \int \frac{2\sec\theta}{\frac{64}{27}\tan^6\theta}\left(\tfrac{2}{\sqrt{3}}\sec^2\theta\right)d\theta = \frac{27}{16\sqrt{3}}\int \frac{\sec^3\theta}{\tan^6\theta}\,d\theta = \frac{27}{16\sqrt{3}}\int \frac{\cos^3\theta}{\sin^6\theta}\,d\theta$$

$$= \frac{27}{16\sqrt{3}}\int \frac{1-\sin^2\theta}{\sin^6\theta}\cos\theta\,d\theta = \boxed{\frac{27}{16\sqrt{3}}\left(\frac{1}{3\sin^3\theta} - \frac{1}{5\sin^5\theta}\right)+c}$$

From a right triangle we get that $\sin\theta = \frac{\sqrt{3}y}{\sqrt{4+3y^2}}$ and so the integral becomes,

$$\int \frac{\sqrt{4+3y^2}}{y^6}\,dy = \frac{27}{16\sqrt{3}}\left(\frac{\left(4+3y^2\right)^{\frac{3}{2}}}{9\sqrt{3}y^3} - \frac{\left(4+3y^2\right)^{\frac{5}{2}}}{45\sqrt{3}y^5}\right)+c = \boxed{\frac{\left(4+3y^2\right)^{\frac{3}{2}}}{16y^3} - \frac{\left(4+3y^2\right)^{\frac{5}{2}}}{80y^5}+c}$$

2. (3 pts) $z = 3\sec\theta \qquad dz = 3\sec\theta\tan\theta\,d\theta \qquad \sqrt{z^2-9} = \sqrt{9\sec^2\theta - 9} = 3\left|\tan\theta\right|$

$$z=4: \quad 4 = 3\sec\theta \quad \rightarrow \quad \sec\theta = \tfrac{4}{3} \quad \Rightarrow \quad \theta = 0.7227$$
$$z=3: \quad 3 = 3\sec\theta \quad \rightarrow \quad \sec\theta = 1 \quad \Rightarrow \quad \theta = 0$$

The range of θ is in the first quadrant ($\frac{\pi}{2} = 1.5708$) and so tangent will be positive and we can then drop the absolute value bars. The integral is then,

$$\int_3^4 z^3\sqrt{z^2-9}\,dz = \int_0^{0.7227} 27\sec^3\theta\left(3\tan\theta\right)3\sec\theta\tan\theta\,d\theta = 243\int_0^{0.7227}\sec^4\theta\tan^2\theta\,d\theta$$

$$= 243\int_0^{0.7227}\left(\tan^2\theta+1\right)\tan^2\theta\sec^2\theta\,d\theta = 243\int_0^{0.7227} u^4+u^2\,du$$

$$= 243\left(\tfrac{1}{5}\tan^5\theta + \tfrac{1}{3}\tan^3\theta\right)\Big|_0^{0.7227} = \boxed{81.4687}$$

6. (2 pts)

$$\frac{2+7x}{(x-4)(4x+1)} = \frac{A}{x-4} + \frac{B}{4x+1} \qquad \Rightarrow \qquad 2+7x = A(4x+1) + B(x-4)$$

$$\begin{array}{lll} x=4: & 30 = A(17) & A = \frac{30}{17} \\ x=-\frac{1}{4}: & \frac{1}{4} = B\left(-\frac{17}{4}\right) \Rightarrow & B = -\frac{1}{17} \end{array}$$

$$\int \frac{2+7x}{(x-4)(4x+1)}\,dx = \int \frac{\frac{30}{17}}{x-4} - \frac{\frac{1}{17}}{4x+1} = \boxed{\tfrac{30}{17}\ln\left|x-4\right| - \tfrac{1}{68}\ln\left|4x+1\right|}$$

8. (3 pts)

$$\frac{z^2-4z}{(z+2)(z^2+25)}=\frac{A}{z+2}+\frac{Bx+C}{x^2+25} \qquad \Rightarrow \qquad z^2-4z=A(z^2+25)+(Bz+C)(z+2)$$

$$=(A+B)z^2+(C+2B)w+25A+2C$$

$$\begin{array}{lll} z^2: & A+B=1 & & A=\tfrac{12}{29} \\ z^1: & C+2B=-4 & \Rightarrow & B=\tfrac{17}{29} \\ z^0: & 25A+2C=0 & & C=-\tfrac{150}{29} \end{array}$$

$$\int\frac{z^2-4z}{(z+2)(z^2+25)}\,dz=\int\frac{\frac{12}{29}}{z+2}+\frac{\frac{17}{29}z-\frac{150}{29}}{z^2+25}\,dz=\frac{1}{29}\int\frac{12}{z+2}+\frac{17z}{z^2+25}-\frac{150}{z^2+25}\,dz$$

$$=\boxed{\tfrac{1}{29}\left[12\ln|z+2|+\tfrac{17}{2}\ln(z^2+25)-30\tan^{-1}\left(\tfrac{z}{5}\right)\right]+c}$$

Not Graded

3. First complete the square on the quadratic under the radical.

$$2-2x-x^2=-(x^2+2x-2)=-(x^2+2x+1-1-2)=-\left[(x+1)^2-3\right]=3-(x+1)^2$$

The integral is then,

$$\int\frac{(x+1)^2}{\left(3-(x+1)^2\right)^{\frac{3}{2}}}\,dx$$

$$x+1=\sqrt{3}\sin\theta \qquad dx=\sqrt{3}\cos\theta\,d\theta \qquad \left(3-(x+1)^2\right)^{\frac{3}{2}}=\left(3-3\sin^2\theta\right)^{\frac{3}{2}}=3\sqrt{3}\left|\cos\theta\right|^3=3\sqrt{3}\cos^3\theta$$

$$\int\frac{(x+1)^2}{\left(3-(x+1)^2\right)^{\frac{3}{2}}}\,dx=\int\frac{3\sin^2\theta}{3\sqrt{3}\cos^3\theta}\sqrt{3}\cos\theta\,d\theta=\int\tan^2\theta\,d\theta=\int\sec^2\theta-1\,d\theta=\tan\theta-\theta+c$$

Using the substitution we can see that we can use $\theta=\sin^{-1}\left(\frac{x+1}{\sqrt{3}}\right)$ and from a right triangle we get $\tan\theta=\frac{x+1}{\sqrt{3-(x+1)^2}}$. The integral is then,

$$\int\frac{(x+1)^2}{\left(3-(x+1)^2\right)^{\frac{3}{2}}}\,dx=\boxed{\frac{x+1}{\sqrt{3-(x+1)^2}}-\sin^{-1}\left(\tfrac{x+1}{\sqrt{3}}\right)+c}$$

4. This is easier than it looks. The trick is to notice that $\mathbf{e}^{4t} = \left(\mathbf{e}^{2t}\right)^2$ so we can use the substitution $u = \mathbf{e}^{2t}$ to reduce this to a trig substitution. I'll combine this substitution and the trig substitution.

$$\mathbf{e}^{2t} = \sin\theta \qquad 2\mathbf{e}^{2t}\,dt = \cos\theta\,d\theta \qquad \sqrt{1-\mathbf{e}^{2t}} = \sqrt{1-\sin^2\theta} = \left|\cos\theta\right| = \cos\theta$$

The integral is then,

$$\int \mathbf{e}^{2t}\sqrt{1-\mathbf{e}^{4t}}\,dt = \int \sqrt{1-\mathbf{e}^{4t}}\,\mathbf{e}^{2t}\,dt = \int \cos\theta\left(\tfrac{1}{2}\cos\theta\right)d\theta = \tfrac{1}{2}\int \cos^2\theta\,d\theta$$
$$= \tfrac{1}{4}\int 1+\cos\left(2\theta\right)d\theta = \tfrac{1}{4}\left(\theta + \tfrac{1}{2}\sin\left(2\theta\right)\right) + c = \tfrac{1}{4}\left(\theta + \sin\theta\cos\theta\right) + c$$

Note that we have to rewrite $\sin\left(2\theta\right)$ using the double angle formula in order to go back to *t*'s. From the substitution we see that we can use $\theta = \sin^{-1}\left(\mathbf{e}^{2t}\right)$ and from a quick right triangle we see that $\cos\theta = \sqrt{1-\mathbf{e}^{4t}}$ and so the integral is then,

$$\int \mathbf{e}^{2t}\sqrt{1-\mathbf{e}^{4t}}\,dt = \boxed{\tfrac{1}{4}\left(\sin^{-1}\left(\mathbf{e}^{2t}\right) + \mathbf{e}^{2t}\sqrt{1-\mathbf{e}^{4t}}\right) + c}$$

5. $$\boxed{f\left(x\right) = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x^5} + \frac{F}{5-9x} + \frac{Gx+H}{x^2-9x+1} + \frac{Ix+J}{\left(x^2-9x+1\right)^2}}$$

7.

$$\frac{w}{\left(3-w\right)\left(w+1\right)^2} = \frac{A}{3-w} + \frac{B}{w+1} + \frac{C}{\left(w+1\right)^2} \qquad \Rightarrow \qquad w = A\left(w+1\right)^2 + B\left(3-w\right)\left(w+1\right) + C\left(3-w\right)$$

$$\begin{array}{lll} w=-1: & -1 = 4C & \\ w=3: & 3 = 16A & \Rightarrow \\ w=0: & 0 = A+3B+3C & \end{array} \qquad \begin{array}{l} A = \tfrac{3}{16} \\ B = \tfrac{3}{16} \\ C = -\tfrac{1}{4} \end{array}$$

$$\int_{-2}^{-3} \frac{w}{\left(3-w\right)\left(w+1\right)^2}\,dw = \int_{-2}^{-3} \frac{\frac{3}{16}}{3-w} + \frac{\frac{3}{16}}{w+1} - \frac{\frac{1}{4}}{\left(w+1\right)^2}\,dw = \left(-\tfrac{3}{16}\ln\left|3-w\right| + \tfrac{3}{16}\ln\left|w+1\right| + \frac{\frac{1}{4}}{w+1}\right)\Bigg|_{-2}^{-3}$$
$$= \boxed{\tfrac{1}{8} + \tfrac{3}{16}\left(\ln 5 + \ln 2 - \ln 6\right) = 0.22078}$$

9. First do long division on the integrand to get,

$$\frac{x^6+x-1}{x^4+4x^2} = x^2 - 4 + \frac{16x^2+x-1}{x^2\left(x^2+4\right)}$$

We now need to partial fraction the third term.

$$\frac{16x^2+x-1}{x^2\left(x^2+4\right)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+4} \quad\Rightarrow\quad 16x^2+x-1=Ax\left(x^2+4\right)+B\left(x^2+4\right)+x^2\left(Cx+D\right)$$

$$=\left(A+C\right)x^3+\left(B+D\right)x^2+4Ax+4B$$

$$\begin{aligned} x^3 &: \quad A+C=0 \\ x^2 &: \quad B+D=16 \\ x^1 &: \quad 4A=1 \\ x^0 &: \quad 4B=-1 \end{aligned} \quad\Rightarrow\quad \begin{aligned} A&=\tfrac{1}{4} \\ B&=-\tfrac{1}{4} \\ C&=-\tfrac{1}{4} \\ D&=\tfrac{65}{4} \end{aligned}$$

The integral is then,

$$\int \frac{x^6+x-1}{x^2\left(x^2+4\right)}\,dx=\int x^2-4+\frac{\frac{1}{4}}{x}-\frac{\frac{1}{4}}{x^2}+\frac{-\frac{1}{4}x+\frac{65}{4}}{x^2+4}\,dx=\int x^2-4+\frac{\frac{1}{4}}{x}-\frac{\frac{1}{4}}{x^2}-\frac{\frac{1}{4}x}{x^2+4}+\frac{\frac{65}{4}}{x^2+4}\,dx$$

$$=\boxed{\tfrac{1}{3}x^3-4x+\tfrac{1}{4}\ln\left|x\right|+\tfrac{1}{4}\tfrac{1}{x}-\tfrac{1}{8}\ln\left(x^2+4\right)+\tfrac{65}{8}\tan^{-1}\left(\tfrac{x}{2}\right)+c}$$