Math 2414 **Homework Set 3 – Solutions** **10 Points**

1. (2 pts) A quick rewrite of the integral and this one is pretty simple.

$$\int \frac{\tan^6(4w)}{\cos^4(4w)}\,dw = \int \sec^4(4w)\tan^6(4w)\,dw = \int \left(1+\tan^2(4w)\right)\tan^6(4w)\sec^2(4w)\,dw$$

$$= \tfrac{1}{4}\int \left(1+u^2\right)u^6\,du = \boxed{\tfrac{1}{4}\left(\tfrac{1}{7}\tan^7(4w)+\tfrac{1}{9}\tan^9(4w)\right)+c}$$

3. (2 pts) With the substitution $u=\sec(t)$ the integral becomes,

$$\int \frac{\sec^2(t)\tan(t)}{\sec^2(t)-\sec(t)-12}\,dt = \int \frac{u}{u^2-u-12}\,du = \int \frac{u}{(u-4)(u+3)}\,du$$

This can be done with partial fractions (I'll leave it to you to check the details of the partial fractions).

$$\int \frac{\sec^2(t)\tan(t)}{\sec^2(t)-\sec(t)-12}\,dt = \int \frac{\frac{3}{7}}{u+3}+\frac{\frac{4}{7}}{u-4}\,du = \boxed{\tfrac{3}{7}\ln\left|\sec(t)+3\right|+\tfrac{4}{7}\ln\left|\sec(t)-4\right|+c}$$

6. (2 pts) This integral will require integration by parts so let's do that before dealing with the limits.

$$u=x^2 \qquad du=2x\,dx \qquad dv = x\mathbf{e}^{-x^2}dx \qquad v=-\tfrac{1}{2}\mathbf{e}^{-x^2}$$

$$\int x^3\mathbf{e}^{-x^2}\,dx = -\tfrac{1}{2}x^2\mathbf{e}^{-x^2}+\int x\mathbf{e}^{-x^2}\,dx = \underline{-\tfrac{1}{2}x^2\mathbf{e}^{-x^2}-\tfrac{1}{2}\mathbf{e}^{-x^2}+c}$$

Now the integral with limits. Note that we'll need to break up the integral and we'll break it up at $x=0$

$$\int_{-\infty}^{\infty} x^2\mathbf{e}^{-x^2}\,dx = \int_{-\infty}^{0} x^2\mathbf{e}^{-x^2}\,dx + \int_{0}^{\infty} x^2\mathbf{e}^{-x^2}\,dx = \lim_{s\to-\infty}\int_s^0 x^2\mathbf{e}^{-x^2}\,dx + \lim_{t\to\infty}\int_0^t x^2\mathbf{e}^{-x^2}\,dx$$

$$= \lim_{s\to-\infty} -\tfrac{1}{2}\mathbf{e}^{-x^2}\left(x^2+1\right)\Big|_s^0 + \lim_{t\to\infty} -\tfrac{1}{2}\mathbf{e}^{-x^2}\left(x^2+1\right)\Big|_0^t$$

$$= \lim_{s\to-\infty}\tfrac{1}{2}\left(\mathbf{e}^{-s^2}\left(s^2+1\right)-1\right)+\lim_{t\to\infty}\tfrac{1}{2}\left(1-\mathbf{e}^{-t^2}\left(t^2+1\right)\right) = -\tfrac{1}{2}+\tfrac{1}{2} = \boxed{0}$$

So, because both of the integrals are convergent the overall integral is **convergent** and its value is **0**.

8. (2 pts) This integral will require integration by parts so let's do that before dealing with the limits.

$$u=\ln\left(\tfrac{1}{y}\right) \qquad du=-\tfrac{1}{y}dy \qquad dv=y^3dy \qquad v=\tfrac{1}{4}y^4$$

$$\int y^3\ln\left(\tfrac{1}{y}\right)dy = \tfrac{1}{4}y^4\ln\left(\tfrac{1}{y}\right)+\tfrac{1}{4}\int y^3\,dy = \tfrac{1}{4}y^4\ln\left(\tfrac{1}{y}\right)+\tfrac{1}{16}y^4+c$$

Now, this is a mix of the two cases we saw in class, so split up the integral, we'll use $y=1$, then take care of the limits for each.

$$\int_0^{\infty} y^3\ln\left(\tfrac{1}{y}\right)dy = \int_0^1 y^3\ln\left(\tfrac{1}{y}\right)dy + \int_1^{\infty} y^3\ln\left(\tfrac{1}{y}\right)dy = \lim_{s\to0^+}\int_s^1 y^3\ln\left(\tfrac{1}{y}\right)dy + \lim_{t\to\infty}\int_1^t y^3\ln\left(\tfrac{1}{y}\right)dy$$

$$= \lim_{s\to0^+}\tfrac{1}{16}y^4\left(1+3\ln\left(\tfrac{1}{y}\right)\right)\Big|_s^1 + \lim_{t\to\infty}\tfrac{1}{16}y^4\left(1+3\ln\left(\tfrac{1}{y}\right)\right)\Big|_1^t$$

$$= \lim_{s\to0^+}\left[\tfrac{1}{16}-\tfrac{1}{16}s^4\left(1+3\ln\left(\tfrac{1}{s}\right)\right)\right]+\lim_{t\to\infty}\left[\tfrac{1}{16}t^4\left(1+3\ln\left(\tfrac{1}{t}\right)\right)-\tfrac{1}{16}\right]$$

The second limit is,

$$\lim_{t\to\infty}\left[\tfrac{1}{16}t^4\left(1+3\ln\left(\tfrac{1}{t}\right)\right)-\tfrac{1}{16}\right]=\left(\infty\right)\left(-\infty\right)-\tfrac{1}{16}=-\infty$$

and so the second integral is divergent therefore the overall limit is **divergent.**

While we don't need to do it the first limit will require L'Hospitals Rule so let's do that for practice.

$$\lim_{s\to 0^+}\left[\tfrac{1}{16}-\tfrac{1}{16}s^4\left(1+3\ln\left(\tfrac{1}{s}\right)\right)\right]=\tfrac{1}{16}-\tfrac{1}{16}\lim_{s\to 0^+}\frac{1+3\ln\left(\frac{1}{s}\right)}{\frac{1}{s^4}}=\tfrac{1}{16}-\tfrac{1}{16}\lim_{s\to 0^+}\frac{-\frac{3}{s}}{-\frac{4}{s^5}}=\tfrac{1}{16}$$

So, the first integral is convergent, but that doesn't make the overall integral convergent. It is still divergent because the second integral is divergent.

10. (2 pts) This looks like it will converge so we want a larger function that we know converges.

$$\frac{\left(x-6\right)^3}{x^8+4\mathbf{e}^{-x}}\le\frac{x^3}{x^8+4\mathbf{e}^{-x}} \qquad \text{b/c}\ \left(x-6\right)^3\le x^3$$

$$\le\frac{x^3}{x^8+0}=\frac{1}{x^5} \qquad \text{b/c}\ 4\mathbf{e}^{-x}\ge 0$$

We know that $\int_{10}^{\infty}\frac{1}{x^5}\,dx$ converges and so be the Comparison Test the original integral must **converge**.

Not Graded

2. Just square the term and integrate each term with appropriate methods (integrations by parts for the second term and trig formulas for the third term).

$$\int\left(x+\cos\left(x\right)\right)^2dx=\int x^2+2x\cos\left(x\right)+\cos^2\left(x\right)dx=\int x^2+\tfrac{1}{2}\left(1+\cos\left(2x\right)\right)dx+\int 2x\cos\left(x\right)dx$$

$$=\int x^2+\tfrac{1}{2}+\tfrac{1}{2}\cos\left(2x\right)dx+2x\sin\left(x\right)-2\int\sin\left(x\right)dx$$

$$=\boxed{\tfrac{1}{3}x^3+\tfrac{1}{2}x+\tfrac{1}{4}\sin\left(2x\right)+2x\sin\left(x\right)+2\cos\left(x\right)+c}$$

4. This is a trig substitution.

$$z=2\tan\theta \qquad dz=2\sec^2\theta\,d\theta \qquad \sqrt{4+z^2}=\sqrt{4+4\tan^2\theta}=2\left|\sec\theta\right|=2\sec\theta$$

$$\int\left(1+z\right)\sqrt{4+z^2}\,dz=\int\left(1+2\tan\theta\right)\left(2\sec\theta\right)\left(2\sec^2\theta\right)d\theta=4\int\sec^3\theta+2\tan\theta\sec^3\theta\,d\theta$$

$$=4\int\sec^3\theta\,d\theta+8\int\sec^2\theta\tan\theta\sec\theta\,d\theta$$

$$=2\left(\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right)+\tfrac{8}{3}\sec^3\theta+c$$

The substitution and a quick right triangle tells us that $\tan\theta = \frac{z}{2}$ and $\sec\theta = \frac{1}{2}\sqrt{4+z^2}$. The integral is,

$$\int (1+z)\sqrt{4+z^2}\,dz = \boxed{\tfrac{1}{2}z\sqrt{4+z^2} + 2\ln\left|\tfrac{z+\sqrt{4+z^2}}{2}\right| + \tfrac{1}{3}\left(4+z^2\right)^{\frac{3}{2}} + c}$$

5. $u = x^{12} \qquad du = 12x^{11}dx \qquad dv = x^{11}\mathbf{e}^{1+x^{12}}dx \qquad v = \frac{1}{12}\mathbf{e}^{1+x^{12}}$

$$\int x^{23}\mathbf{e}^{1+x^{12}}\,dx = \tfrac{1}{12}x^{12}\mathbf{e}^{1+x^{12}} - \int x^{11}\mathbf{e}^{1+x^{12}}\,dx = \tfrac{1}{12}x^{12}\mathbf{e}^{1+x^{12}} - \tfrac{1}{12}\mathbf{e}^{1+x^{12}} + c$$

7. This integral will require partial fractions so that will need to be done prior to dealing with the limits. The partial fractions work is pretty simple so I'll leave it to you to verify the work.

$$\int \frac{6}{z^2-4z-12}\,dz = \int \frac{\frac{3}{4}}{z-6} - \frac{\frac{3}{4}}{z+2}\,dz = \tfrac{3}{4}\ln|z-6| - \tfrac{3}{4}\ln|z+2| + c$$

We have division by zero at $z=-2$ so we'll need to break up the integral there and then take care of the limits.

$$\begin{aligned}
\int_{-3}^{1}\frac{6}{z^2-4z-12}\,dz &= \int_{-3}^{-2}\frac{6}{z^2-4z-12}\,dz + \int_{-2}^{1}\frac{6}{z^2-4z-12}\,dz \\
&= \lim_{s\to -2^-}\int_{-3}^{s}\frac{6}{z^2-4z-12}\,dz + \lim_{t\to -2^+}\int_{t}^{1}\frac{6}{z^2-4z-12}\,dz \\
&= \lim_{s\to -2^-}\left(\tfrac{3}{4}\ln|z-6| - \tfrac{3}{4}\ln|z+2|\right)\Big|_{-3}^{s} + \lim_{t\to -2^+}\left(\tfrac{3}{4}\ln|z-6| - \tfrac{3}{4}\ln|z+2|\right)\Big|_{t}^{1} \\
&= \lim_{s\to -2^-}\left(\tfrac{3}{4}\ln|s-6| - \tfrac{3}{4}\ln|s+2| - \tfrac{3}{4}\ln 9\right) \\
&\qquad + \lim_{t\to -2^+}\left(\tfrac{3}{4}\ln 5 - \tfrac{3}{4}\ln 3 - \tfrac{3}{4}\ln|t-6| + \tfrac{3}{4}\ln|t+2|\right) \\
&= \left(\tfrac{3}{4}\ln 8 - \tfrac{3}{4}\ln 9 + \infty\right) + \left(\tfrac{3}{4}\ln 5 - \tfrac{3}{4}\ln 3 - \tfrac{3}{4}\ln 8 - \infty\right)
\end{aligned}$$

So both of the integrals are divergent and so the integral is **divergent.**

9. This looks like it will diverge so let's find a smaller function that we know diverges.

$$\begin{aligned}
\frac{x^4+\sin^4(4x)}{x^5\cos^2(2-x)} &\ge \frac{x^4+0}{x^5\cos^2(2-x)} && \text{b/c } \sin^4(4x)\ge 0 \\
&\ge \frac{x^4}{x^5(1)} = \frac{1}{x} && \text{b/c } \cos^2(2-x)\le 1
\end{aligned}$$

We know that $\int_1^\infty \frac{1}{x}\,dx$ diverges and so be the Comparison Test the original integral must **diverge**.

11. For this problem we have $f(x) = \ln\left(1+\mathbf{e}^{\cos(x)}\right)$ and $\Delta x = \frac{1}{2}$.

MidPoint Rule

$$\int_{-1}^{1} \ln\left(1+\mathbf{e}^{\cos(x)}\right)dx \approx \tfrac{1}{2}\left[f\left(-0.75\right)+f\left(-0.25\right)+f\left(0.25\right)+f\left(0.75\right)\right]=2.4151$$

Trapezoid Rule

$$\int_{-1}^{1} \ln\left(1+\mathbf{e}^{\cos(x)}\right)dx \approx \tfrac{1}{4}\left[f\left(-1\right)+2f\left(-0.5\right)+2f\left(0\right)+2f\left(0.5\right)+f\left(1\right)\right]=2.38158$$

Simpson's Rule

$$\int_{-1}^{1} \ln\left(1+\mathbf{e}^{\cos(x)}\right)dx \approx \tfrac{1}{6}\left[f\left(-1\right)+4f\left(-0.5\right)+2f\left(0\right)+4f\left(0.5\right)+f\left(1\right)\right]=2.40456$$

While I didn't ask for it, here is the exact value for comparison's sake,

$$\int_{-1}^{1} \ln\left(1+\mathbf{e}^{\cos(x)}\right)dx = 2.40389$$