2. (2 pts) The terms are clearly positive and increasing *n* **only** increases the denominator and so the terms will decrease. Therefore the integral test can be used. The integral will involve partial fractions and I'll leave it to you to verify the partial fraction details.

$$\int_1^\infty \frac{1}{x^2+4x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\frac{1}{4}}{x} - \frac{\frac{1}{4}}{x+4}\,dx = \lim_{t\to\infty}\tfrac{1}{4}\left(\ln|x| - \ln|x+4|\right)\Big|_1^t = \lim_{t\to\infty}\tfrac{1}{4}\left(\ln\left(\frac{t}{t+4}\right)+\ln(5)\right) = \ln(5)$$

The integral converges and so by the **Integral Test** the series will **converge**.

5. (2 pts) This looks like it will diverge so we'll need a smaller function we can prove diverges.

$$\frac{n^4+3\sin^4(6n)}{n^5\cos^2(5n+1)} \ge \frac{n^4+3(0)}{n^5\cos^2(5n+1)} \qquad \text{b/c } 0\le \sin^4(6n)\le 1$$

$$\ge \frac{n^4}{n^5(1)} = \frac{1}{n} \qquad \text{b/c } 0\le \cos^2(5n+1)\le 1$$

Now, $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges and so by the **Comparison Test** the original series will **diverge**.

7. (2 pts) We'll use the limit comparison test with $\sum_{n=1}^{\infty}\frac{1}{n}$ as the second series and note that this diverges.

$$c = \lim_{n\to\infty}\frac{2n^6-n^4+n^2}{5n^7+6n^3-4}\,\frac{n}{1} = \lim_{n\to\infty}\frac{n^7\left(2-\frac{1}{n^2}+\frac{1}{n^4}\right)}{n^7\left(5+\frac{6}{n^4}-\frac{4}{n^7}\right)} = \frac{2}{5}$$

So, $0 < c = \frac{2}{5} < \infty$ and so by the **Limit Comparison Test** both series will have the same convergence. The second series diverges and so the original series will **diverges**.

9. (2 pts) In this case we have,

$$b_n = \frac{4-n}{1-2n^2} > 0 \qquad\qquad \lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{4-n}{1-2n^2} = 0$$

We'll need to do a little Calc I for the decreasing part here.

$$f(x) = \frac{4-x}{1-2x^2} \qquad\qquad f'(x) = \frac{-2x^2+16x-1}{\left(1-2x^2\right)^2}$$

The critical points are $x = 0.0630$ and $x = 7.9370$. From the increasing/decreasing information (I'll leave it to you to verify this) we can see that the function , and hence the series terms, will increase in the range $4 \le n < 7.937$ and decrease in the range $n > 7.937$ and so the series terms will be decreasing eventually. Therefore, by the **Alternating Series Test** the series will **converge**.

11. (2 pts) Using the Ratio Test we get,

$$L = \lim_{n\to\infty}\left|\frac{(3n+4)!}{(n-1)!}\frac{(n-2)!}{(3n+1)!}\right| = \lim_{n\to\infty}\left|\frac{(3n+4)(3n+3)(3n+2)(3n+1)!}{(n-1)(n-2)!}\frac{(n-2)!}{(3n+1)!}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(3n+4)(3n+3)(3n+2)}{(n-1)}\right| = \infty > 1$$

By the Ratio Test the series will **diverge.**

Not Graded

1. We can use the integral test or a quick rewrite shows that $p = \frac{7}{12} < 1$ and so the series will **diverge**.

$$\sum_{n=1}^{\infty}\frac{8}{n^{\frac{2}{6}}\,n^{\frac{1}{4}}} = \sum_{n=1}^{\infty}\frac{8}{n^{\frac{7}{12}}}$$

3. First notice that for $1 \le n < \infty$ we have $0 < \frac{1}{n} \le 1$ and in this range inverse tangent is positive. Let's next take a derivative to check for decreasing.

$$f(x) = \tan^{-1}\left(\tfrac{1}{x}\right) \qquad f'(x) = \frac{-\frac{1}{x^2}}{1+\frac{1}{x^2}}$$

So, in the range $1 \le x < \infty$ inverse tangent is decreasing and so the series terms are decreasing and we can then use the integral test. We did an integral similar to this on the first homework set so I'll leave it to you to verify the most of the integration by parts work.

$$\int \tan^{-1}\left(\tfrac{1}{x}\right)dx = x\tan^{-1}\left(\tfrac{1}{x}\right) + \int \frac{\frac{1}{x}}{1+\frac{1}{x^2}}dx = x\tan^{-1}\left(\tfrac{1}{x}\right) + \int \frac{x}{x^2+1}dx = x\tan^{-1}\left(\tfrac{1}{x}\right) + \ln\left(x^2+1\right) + c$$

$$\int_1^{\infty}\tan^{-1}\left(\tfrac{1}{x}\right)dx = \lim_{t\to\infty}\int_1^t \tan^{-1}\left(\tfrac{1}{x}\right)dx = \lim_{t\to\infty}\left(x\tan^{-1}\left(\tfrac{1}{x}\right) + \ln\left(x^2+1\right)\right)\Big|_1^t$$
$$= \lim_{t\to\infty}\left(t\tan^{-1}\left(\tfrac{1}{t}\right) + \ln\left(t^2+1\right) - \left(\tan^{-1}(1) + \ln(2)\right)\right)$$
$$= 1 + \infty - \left(\tan^{-1}(1) + \ln(2)\right) = \infty$$

The integral diverges and so by the **Integral Test** the series will **diverge**. You were able do the first limit right?

$$\lim_{t\to\infty} t\tan^{-1}\left(\tfrac{1}{t}\right) = \lim_{t\to\infty}\frac{\tan^{-1}\left(\frac{1}{t}\right)}{\frac{1}{t}} = \lim_{t\to\infty}\frac{\frac{-\frac{1}{t^2}}{1+\frac{1}{t^2}}}{-\frac{1}{t^2}} = \lim_{t\to\infty}\frac{1}{1+\frac{1}{t^2}} = 1$$

4. This looks like it will converge so we'll need a larger function we can prove converges.

$$\frac{3-\cos^2(3n)}{n^2+\mathbf{e}^{-n}} \le \frac{3-0}{n^2+\mathbf{e}^{-n}} \qquad \text{b/c } 0 \le 3-\cos^2(3n) \le 3$$

$$\le \frac{3}{n^2+0} = \frac{3}{n^2} \qquad \text{b/c } 0 < \mathbf{e}^{-n} < 1$$

Now, $\sum_{n=2}^{\infty} \frac{3}{n^2}$ converges and so by the **Comparison Test** the original series will **converge**.

6. We'll use the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ as the second series and note that this converges.

$$c = \lim_{n\to\infty} \frac{6n^2+2n-1}{\sqrt[3]{n^{12}-n^3}} \frac{n^2}{1} \lim_{n\to\infty} \frac{n^4\left(6+\frac{2}{n}-\frac{1}{n^2}\right)}{n^4\sqrt[3]{1-\frac{1}{n^9}}} = 6$$

So, $0 < c = 6 < \infty$ and so by the **Limit Comparison Test** both series will have the same convergence. The second series converges and so the original series will **converges**.

8. First note that $\cos(n\pi) = (-1)^n$ and so this really is an alternating series with,

$$b_n = \frac{1}{6n^3+9n+2} \qquad \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{6n^3+9n+2} = 0$$

Also, increasing *n* only increases the denominator and so the b_n are decreasing and so by the **Alternating Series Test** the series will **converge**.

10. In this case we have,

$$b_n = \frac{n+4}{7n+1} \qquad \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n+4}{7n+1} = \frac{1}{7}$$

So, the Alternating series test will not work for this series. The Divergence Test can then be used.

$$\lim_{n\to\infty} \frac{(-1)^{n+5}(n+4)}{7n+1} = \left[\lim_{n\to\infty}(-1)^{n+5}\right]\left[\lim_{n\to\infty}\frac{n+4}{7n+1}\right] = \left[\lim_{n\to\infty}(-1)^{n+5}\right]\left[\frac{1}{7}\right] \text{ - Does Not Exist}$$

By the **Divergence Test** this series will **diverge**.

12. Using the Ratio Test we get,

$$L = \lim_{n\to\infty}\left|\frac{9^{-n}\,2^{4+3n}}{(n+1)^2}\frac{n^2}{9^{1-n}\,2^{1+3n}}\right| = \lim_{n\to\infty}\left|\frac{2^3}{(n+1)^2}\frac{n^2}{9^1}\right| = \lim_{n\to\infty}\left|\frac{8n^2}{9(n+1)^2}\right| = \frac{8}{9} < 1$$

By the Ratio Test the series will **converge.**

13. By the Root Test the series will **converge :** $L = \lim_{n\to\infty}\left|\frac{6^{3+2n}}{n^{n+2}}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{6^{\frac{3}{n}+2}}{n^{1+\frac{2}{n}}} = \frac{6^2}{\infty} = 0 < 1$

14. By the Root Test the series will **converge**.

$$L=\lim_{n\to\infty}\left|\left(\frac{6-8n}{1+11n}\right)^{3n}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\frac{6-8n}{1+11n}\right|^{3}=\left|\frac{-8}{11}\right|^{3}=\frac{512}{1331}<1$$

15. Do not make the mistake of saying that this series is a harmonic series and hence diverges. Because there are two of them here it is not a harmonic series. This is a telescoping series so we'll need to set up the partial sums.

$$s_N=\sum_{n=2}^{N}\frac{1}{n}-\frac{1}{n-1}=\left(\frac{1}{2}-\frac{1}{1}\right)+\left(\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{4}-\frac{1}{3}\right)+\cdots$$

$$\cdots+\left(\frac{1}{N-2}-\frac{1}{N-3}\right)+\left(\frac{1}{N-1}-\frac{1}{N-2}\right)+\left(\frac{1}{N}-\frac{1}{N-1}\right)$$

$$=-1+\frac{1}{N}\qquad\qquad \lim_{n\to\infty}s_N=\lim_{n\to\infty}\left(-1+\frac{1}{N}\right)=-1$$

The sequence of partial sums converges and so the series **converges** (and has a value of -1).

16. Note that this is an alternating series as the alternating sign can be in either the numerator or denominator. We can always move it to the numerator if we want as follows.

$$\sum_{n=1}^{\infty}\frac{4}{(-1)^{n+2}(n+3)}\frac{(-1)^{n+2}}{(-1)^{n+2}}\sum_{n=1}^{\infty}\frac{4(-1)^{n+2}}{n+3}$$

So,

$$b_n=\frac{4}{n+3}\qquad\qquad \lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{4}{n+3}=0$$

Increasing n only increases the denominator and so these will decrease and so by the **Alternating Series Test** this series will **converge**.

17. $L=\lim_{n\to\infty}\left|\frac{(n+1)^2}{3^{-1-2n}(-4)^{5+n}}\frac{3^{1-2n}(-4)^{4+n}}{n^2}\right|=\lim_{n\to\infty}\left|\frac{(n+1)^2}{(-4)^1}\frac{3^2}{n^2}\right|=\frac{9}{4}\lim_{n\to\infty}\frac{(n+1)^2}{n^2}=\frac{9}{4}>1$

18. These terms are positive and if we do some Calc I we get,

$$f(x)=\frac{[\ln x]^2}{x}\qquad f'(x)=\frac{2\ln x-[\ln x]^2}{x^2}=\frac{\ln x(2-\ln x)}{x^2}\quad\to\quad \ln x=2\quad\to\quad x=\mathbf{e}^2$$

This function has a single critical point as shown above and I'll leave it to you to verify that the derivative will be negative for $x>\mathbf{e}^2$ and so the function will be eventually be decreasing and so we can use the Integral Test on this series.

$$\int_1^{\infty}\frac{[\ln x]^2}{x}dx=\lim_{t\to\infty}\int_1^{t}\frac{[\ln x]^2}{x}dx=\lim_{t\to\infty}\tfrac{1}{3}[\ln x]^3\Big|_1^t=\lim_{t\to\infty}\tfrac{1}{3}[\ln t]^3=\infty$$

So, this integral is divergent and so by the **Integral Test** the series **diverges**.

19. By the **Ration Test** this series will **converge.**

$$L = \lim_{n\to\infty}\left|\frac{2^{n+2}}{(2n+3)!}\frac{(2n+1)!}{2^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2^{n+2}}{(2n+3)(2n+2)(2n+1)!}\frac{(2n+1)!}{2^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2^{1}}{(2n+3)(2n+2)}\right| = 0 < 1$$