**3. (3 pts)** $L = \lim_{n\to\infty} \left| \frac{6^{1+n}(x+3)^n}{2^{2+3n}} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \frac{6^{\frac{1}{n}+1}(x+3)}{2^{\frac{2}{n}+3}} \right| = \frac{6}{8}|x+3|$

$$\tfrac{3}{4}|x+3| < 1 \qquad |x+3| < \tfrac{4}{3} \qquad -\tfrac{4}{3} < x+3 < \tfrac{4}{3} \qquad \rightarrow \qquad -\tfrac{13}{3} < x < -\tfrac{5}{3}$$

Checking the endpoints gives,

$$x = -\tfrac{13}{3}: \sum_{n=1}^{\infty} \frac{6^{1+n}\left(-\frac{4}{3}\right)^n}{2^{2+3n}} = \sum_{n=1}^{\infty} \frac{6\left(2^n 3^n\right)(-1)^n \frac{2^{2n}}{3^n}}{4\left(2^{3n}\right)} = \sum_{n=1}^{\infty} \frac{6(-1)^n}{4} \quad \text{Diverges - Divergence Test}$$

$$x = -\tfrac{5}{3}: \sum_{n=1}^{\infty} \frac{6^{1+n}\left(\frac{4}{3}\right)^n}{2^{2+3n}} = \sum_{n=1}^{\infty} \frac{6\left(2^n 3^n\right) \frac{2^{2n}}{3^n}}{4\left(2^{3n}\right)} = \sum_{n=1}^{\infty} \frac{6}{4} \quad \text{Diverges - Divergence Test}$$

The **Radius of Convergence** is $R = \frac{4}{3}$ and the **Interval of Convergence** is $-\frac{13}{3} < x < -\frac{5}{3}$.

**6. (3 pts)**

$$f(t) = \frac{2t}{3}\frac{1}{1-\frac{5}{3}t^2} = \frac{2t}{3}\sum_{n=0}^{\infty}\left(\tfrac{5}{3}t^2\right)^n = \frac{2t}{3}\sum_{n=0}^{\infty}\tfrac{5^n}{3^n}t^{2n} = \boxed{\sum_{n=0}^{\infty} 2\tfrac{5^n}{3^{n+1}}t^{2n+1}}$$

This series will converge (even though I didn't ask for this) for,

$$\left|\tfrac{5}{3}t^2\right| < 1 \quad \rightarrow \quad |t|^2 < \tfrac{3}{5} \quad \rightarrow \quad |t| < \sqrt{\tfrac{3}{5}}$$

**10. (4 pts)** First get the derivatives and their value at $x = 0$.

$$f(x) = \ln(7+2x) \qquad\qquad f(0) = \ln(7)$$

$$f'(x) = \tfrac{2}{7+2x} = 2(7+2x)^{-1} \qquad\qquad f'(0) = 2(7)^{-1}$$

$$f''(x) = -2^2(7+2x)^{-2} \qquad\qquad f''(0) = -2^2(7)^{-2}$$

$$f'''(x) = 2^3(2)(7+2x)^{-3} \qquad\qquad f'''(0) = 2^3(2)(7)^{-3}$$

$$f^{(4)}(x) = -2^4(2)(3)(7+2x)^{-4} \qquad\qquad f^{(4)}(0) = -2^4(2)(3)(7)^{-4}$$

$$f^{(5)}(x) = 2^5(2)(3)(4)(7+2x)^{-5} \qquad\qquad f^{(5)}(0) = 2^5(2)(3)(4)(7)^{-5}$$

The general formula is,

$$f^{(n)}(x) = \frac{(-1)^{n+1}2^n(n-1)!}{(7+2x)^n}, \quad n \ge 1 \qquad\qquad f^{(n)}(0) = \frac{(-1)^{n+1}2^n(n-1)!}{7^n}, \quad n \ge 1$$

Note that the general formula is NOT valid for $n = 0$. The Taylor series is then,

$$\ln(7+2x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=f(0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$
$$=\ln(7)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n(n-1)!}{7^n\,n!}x^n=\boxed{\ln(7)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n}{7^n\,n}x^n}$$

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***Not Graded***

**1.** $L=\lim_{n\to\infty}\left|\frac{(4x-2)^{n+1}}{3(n+1)}\frac{3n}{(4x-2)^n}\right|=\lim_{n\to\infty}\left|\frac{(4x-2)(3n)}{3(n+1)}\right|=|4x-2|=4\left|x-\tfrac{1}{2}\right|$

$$4\left|x-\tfrac{1}{2}\right|<1 \qquad \left|x-\tfrac{1}{2}\right|<\tfrac{1}{4} \qquad -\tfrac{1}{4}<x-\tfrac{1}{2}<\tfrac{1}{4} \qquad \to \qquad \tfrac{1}{4}<x<\tfrac{3}{4}$$

Checking the endpoints gives,

$x=\frac{1}{4}:\sum_{n=1}^{\infty}\frac{(-1)^n}{3n}$ Converges - Altenating Series Test

$x=\frac{3}{4}:\sum_{n=1}^{\infty}\frac{(1)^n}{3n}=\sum_{n=1}^{\infty}\frac{1}{3n}$ Diverges - Harmonic Series

The **Radius of Convergence** is $R=\frac{1}{4}$ and the **Interval of Convergence** is $\frac{1}{4}\le x<\frac{3}{4}$.

**2.**

$$L=\lim_{n\to\infty}\left|\frac{(2n+2)!(x+5)^{n+1}}{(2n)!(x+5)^n}\right|=\lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)(2n)!(x+5)}{(2n)!}\right|$$
$$=|x+5|\lim_{n\to\infty}(2n+2)(2n+1)=\infty,\quad x\neq-5$$

So, $L=\infty$ provided $x\neq-5$ and so will only converge for $x=-5$. Therefore the **Radius of Convergence** is $R=0$ and the **Interval of Convergence** is $x=-5$.

**4.** $L=\lim_{n\to\infty}\left|\frac{(3x-1)^n}{(3+n)^{3n}}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\frac{3x-1}{(3+n)^3}\right|=0$

So, because $L<1$ for all *x*'s this series will converge for all *x* and so the **Radius of Convergence** is $R=\infty$ and the **Interval of Convergence** is $-\infty<x<\infty$.

**5.** $h(x)=6x^7\frac{1}{1-\left(-4\sqrt{x}\right)}=6x^7\sum_{n=0}^{\infty}\left(-4x^{\frac{1}{2}}\right)^n=\boxed{\sum_{n=0}^{\infty}6(-4)^n x^{\frac{1}{2}n+7}}$

This series will converge (even though I didn't ask for this) for,

$$\left|-4x^{\frac{1}{2}}\right|<1 \quad \to \quad |x|^{\frac{1}{2}}<\tfrac{1}{4} \quad \to \quad |x|<\tfrac{1}{16}$$

**7.** This can be done with the formula we derived in class.

$$f(x)=x^2\sin\left(8x^3\right)=x^2\sum_{n=0}^{\infty}\frac{(-1)^n\left(8x^3\right)^{2n+1}}{(2n+1)!}=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n 8^{2n+1}x^{6n+3}}{(2n+1)!}}$$

**8.** First get some derivatives and their value at $x=7$.

$$\begin{aligned}
g(x)&=8-6x-4x^2 & g(7)&=-230\\
g'(x)&=-6-8x & g'(7)&=-62\\
g''(x)&=-8 & g''(7)&=-8\\
g^{(n)}(x)&=0,\ n=3,4,5,\ldots & g^{(n)}(7)&=0,\ n=3,4,5,\ldots
\end{aligned}$$

The Taylor series is then,

$$8-6x-4x^2=g(7)+g'(7)(x-7)+\tfrac{g''(7)}{2!}(x-7)^2=\boxed{-230-62(x-7)-4(x-7)^2}$$

If you think about it there shouldn't be too much surprise that the Taylor series "terminates". We are writing a 2nd degree polynomial in terms of a 2nd degree polynomial!

**9.** First get some derivatives and their value at $x=-3$.

$$\begin{aligned}
h(x)&=(1-x)^{\frac{1}{2}} & h(-3)&=2\\
h'(x)&=-\tfrac{1}{2}(1-x)^{-\frac{1}{2}} & h'(-3)&=-\tfrac{1}{4}\\
h''(x)&=-\tfrac{1}{2}\left(\tfrac{1}{2}\right)(1-x)^{-\frac{3}{2}} & &\\
h'''(x)&=-\tfrac{1}{2}\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)(1-x)^{-\frac{5}{2}} & &\\
h^{(4)}(x)&=-\tfrac{1}{2}\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)(1-x)^{-\frac{7}{2}} & &\\
&\vdots & &\\
h^{(n)}(x)&=-\tfrac{1(3)(5)\cdots(2n-3)}{2^n}(1-x)^{-\frac{1}{2}(2n-1)},\ n\ge 2 & h^{(n)}(-3)&=-\tfrac{1(3)(5)\cdots(2n-3)}{2^n 2^{2n-1}},\ n\ge 2
\end{aligned}$$

Note that in this case our formula will not work for all *n*. This happens on occasion so don't get excited about it. The Taylor Series is then,

$$\sqrt{1-x} = \sum_{n=0}^{\infty} \frac{h^{(n)}(-3)}{n!}(x+3)^n = \frac{h^{(0)}(-3)}{0!} + \frac{h^{(1)}(-3)}{1!}(x+3) + \sum_{n=2}^{\infty} \frac{h^{(n)}(-3)}{n!}(x+3)^n$$

$$= \boxed{2 - \frac{1}{4}(x+3) + \sum_{n=2}^{\infty} \frac{(-1)(1)(3)(5)\cdots(2n-3)}{n!\, 2^{3n-1}}(x+3)^n}$$