Math 2415 **Homework Set 4** **10 Points**

1. (2 pts) We'll need the gradient of the function : $F(x,y,z) = x^3\cos(4y) + y^2(3-z)^2 = 8$

$$\nabla F(x,y,z) = \left\langle 3x^2\cos(4y), -4x^3\sin(4y) + 4(3-z)^2, -8y(3-z)\right\rangle \qquad \nabla F(-1,0,5) = \langle 3,16,0\rangle$$

The tangent plane is then,

$$3(x+1) + 16(y-0) + (0)(z-5) = 0 \quad \Rightarrow \quad \boxed{3x + 16y = -3}$$

The normal line is,

$$\boxed{\vec{r}(t) = \langle -1,0,5\rangle + t\langle 3,16,0\rangle = \langle -1+3t, 16t, 5\rangle}$$

3. (4 pts) Here are all the derivatives and *D*.

$$h_x = 8x^3 + 4y^2 - 8x \qquad h_y = 8xy - 4y \qquad h_{xx} = 24x^2 - 8$$

$$h_{xy} = 8y \qquad h_{yy} = 8x - 4 \qquad D = (24x^2 - 8)(8x-4) - 64y^2$$

Now, find the critical points.

$$8x^3 + 4y^2 - 8x = 0 \qquad 8xy - 4y = 4y(2x-1) = 0 \quad \Rightarrow \quad x = \tfrac{1}{2},\ y = 0$$

$$x = \tfrac{1}{2} : 4y^2 - 3 = 0 \quad \Rightarrow \quad y = \pm\sqrt{\tfrac{3}{4}} = \pm\tfrac{\sqrt{3}}{2}$$

$$y = 0 : 8x^3 - 8x = 8x(x^2 - 1) = 0 \quad \Rightarrow \quad x = 0,\ x = \pm 1$$

So, it looks like we've got 5 critical points. Here they are and their classifications.

$(0,0)$	:	$D = 32 > 0$	$h_{xx}(0,0) = -8 < 0$	Relative Maximum
$(1,0)$	:	$D = 64 > 0$	$h_{xx}(1,0) = 16 > 0$	Relative Minimum
$(-1,0)$	:	$D = -192 < 0$		Saddle Point
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	:	$D = -48 < 0$		Saddle Point
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	:	$D = -48 < 0$		Saddle Point

8. (4 pts) From the constraint we have $-\sqrt{7} \le x, y, z \le \sqrt{7}$ so we are on a closed, bonded region. Therefore, we know that absolute extrema will exist.

Here are the equations we need to solve and notice that we can't have $\lambda = 0$ as the last would clearly not be valid.

$$-2x = 2x\lambda \quad \Rightarrow \quad x = 0,\ \lambda = -1$$

$$-4y = 2y\lambda \quad \Rightarrow \quad y = 0,\ \lambda = -2$$

$$8 = 2z\lambda \quad \Rightarrow \quad z = \tfrac{4}{\lambda}$$

$$x^2 + y^2 + z^2 = 7$$

The first two equations gives several possibilities. Note however, that regardless of which of those we use the results from the last equation will be true. So, let's start with the two possibilities from the first equation. First, if $x = 0$ we then have either $y = 0$ or $\lambda = -2$ from the second equation. So, going through each we get,

$$x = 0,\ y = 0\ :\ z^2 = 7 \quad \Rightarrow \quad z = \pm\sqrt{7} \quad \Rightarrow \quad \left(0,0,\sqrt{7}\right),\ \left(0,0,-\sqrt{7}\right)$$
$$x = 0,\ \lambda = -2\ :\ z = \tfrac{4}{-2} = -2 \quad \Rightarrow \quad y^2 = 3 \quad \Rightarrow \quad \left(0,\sqrt{3},-2\right),\ \left(0,-\sqrt{3},-2\right)$$

Next, let's see what happens if we let $\lambda = -1$, In this case we get only one possibility from the second equation (can't have λ being two different values….) : $y = 0$. So, in this case we have,

$$\lambda = -1,\ \ y = 0\ :\ z = \tfrac{4}{-1} = -4 \quad \Rightarrow \quad x^2 = -9 \quad \Rightarrow \quad x = \pm 3i$$

We don't deal with complex numbers in this class so we can ignore these values. We therefore have four points to plug into the function,

$$f\left(0,0,\sqrt{7}\right) = 8\sqrt{7} \qquad f\left(0,0,-\sqrt{7}\right) = -8\sqrt{7} \qquad f\left(0,\sqrt{3},-2\right) = f\left(0,-\sqrt{3},-2\right) = -22$$

The minimum is -22 which occurs at $\left(0,\sqrt{3},-2\right)$ and $\left(0,-\sqrt{3},-2\right)$. The maximum is $8\sqrt{7}$ which occurs at $\left(0,0,\sqrt{7}\right)$.

Not Graded

2. The gradient of the function $F\left(x,y,z\right) = 3x^2 - y^2 + 6z^2 = 1$ is $\nabla F = \left\langle 6x,-2y,12z\right\rangle$ and we know that this is orthogonal to the surface at any point and so is the normal vector for the tangent plane. We want the point(s) where this is parallel to the normal vector of the given plane, $\vec{n} = \left\langle 9,-3,1\right\rangle$. We know that parallel vectors will be parallel if they are scalar multiplies of each other. In other words, there is a number *c* so that,

$$\nabla F = c\vec{n} \qquad \Rightarrow \qquad \left\langle 6x,-2y,12z\right\rangle = c\left\langle 9,-3,1\right\rangle = \left\langle 9c,-3c,c\right\rangle$$

Setting components equal gives the following three equations that can be solved for *x*, *y*, and *z*.

$$\begin{aligned} 6x &= 9c \\ -2y &= -3c \\ 12z &= c \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x &= \tfrac{3}{2}c \\ y &= \tfrac{3}{2}c \\ z &= \tfrac{1}{12}c \end{aligned}$$

Now, since the points must be on the surface they must also satisfy $3x^2 - y^2 + 6z^2 = 1$ so the equations above into this and solve for *c*.

$$3\left(\tfrac{3}{2}c\right)^2 - \left(\tfrac{3}{2}c\right)^2 + 6\left(\tfrac{1}{12}c\right)^2 = 1 \qquad \Rightarrow \qquad \tfrac{109}{24}c^2 = 1 \qquad \Rightarrow \qquad c = \pm\sqrt{\tfrac{24}{109}} = \pm 0.4692$$

We've got 2 *c*'s and so that means that we have two possible points. One for each *c*. The points are,

$$\boxed{\left(0.7038, 0.7038, 0.0391\right) \qquad \left(-0.7038, -0.7038, -0.0391\right)}$$

4. First get all the derivatives and *D*. Also after each derivative I factored a little to help with the next.

$$g_x = \left(1-16x^2\right)y\mathbf{e}^{-\left(8x^2+2y^2\right)} \qquad g_y = \left(1-4y^2\right)x\mathbf{e}^{-\left(8x^2+2y^2\right)}$$

$$g_{xx} = 16xy\left(16x^2-3\right)\mathbf{e}^{-\left(8x^2+2y^2\right)} \quad g_{xy} = \left(1-16x^2\right)\left(1-4y^2\right)\mathbf{e}^{-\left(8x^2+2y^2\right)} \quad g_{yy} = 4xy\left(4y^2-3\right)\mathbf{e}^{-\left(8x^2+2y^2\right)}$$

$$D = 64x^2y^2\left(16x^2-3\right)\left(4y^2-3\right)\mathbf{e}^{-2\left(8x^2+2y^2\right)} - \left(1-16x^2\right)^2\left(1-4y^2\right)^2\mathbf{e}^{-2\left(8x^2+2y^2\right)}$$

Kind of messy but there they are. Now find the critical points.

$$\left(1-16x^2\right)y\mathbf{e}^{-\left(8x^2+2y^2\right)} = 0 \qquad \left(1-4y^2\right)x\mathbf{e}^{-\left(8x^2+2y^2\right)} = 0 \qquad \Rightarrow \qquad x = 0, y = \pm\tfrac{1}{2}$$

$$x = 0 \quad : \quad y\mathbf{e}^{-\left(8x^2+2y^2\right)} = 0 \qquad \Rightarrow \qquad y = 0$$

$$y = \tfrac{1}{2} \quad : \quad \tfrac{1}{2}\left(1-16x^2\right)\mathbf{e}^{-\left(8x^2+2y^2\right)} = 0 \quad \Rightarrow \qquad x = \pm\tfrac{1}{4}$$

$$y = -\tfrac{1}{2} : \quad -\tfrac{1}{2}\left(1-16x^2\right)\mathbf{e}^{-\left(8x^2+2y^2\right)} = 0 \Rightarrow \qquad x = \pm\tfrac{1}{4}$$

So, we have 5 critical points. Here they are and their classifications.

$\left(0,0\right)$	:	$D = -1 < 0$		Saddle Point
$\left(\tfrac{1}{4},\tfrac{1}{2}\right)$	:	$D = 4\mathbf{e}^{-2} > 0$	$f_{xx}\left(\tfrac{1}{4},\tfrac{1}{2}\right) = -4\mathbf{e}^{-1} < 0$	Relative Maximum
$\left(-\tfrac{1}{4},\tfrac{1}{2}\right)$	:	$D = 4\mathbf{e}^{-2} > 0$	$f_{xx}\left(-\tfrac{1}{4},\tfrac{1}{2}\right) = 4\mathbf{e}^{-1} > 0$	Relative Minimum
$\left(\tfrac{1}{4},-\tfrac{1}{2}\right)$	:	$D = 4\mathbf{e}^{-2} > 0$	$f_{xx}\left(\tfrac{1}{4},-\tfrac{1}{2}\right) = 4\mathbf{e}^{-1} > 0$	Relative Minimum
$\left(-\tfrac{1}{4},-\tfrac{1}{2}\right)$	:	$D = 4\mathbf{e}^{-2} > 0$	$f_{xx}\left(-\tfrac{1}{4},-\tfrac{1}{2}\right) = -4\mathbf{e}^{-1} < 0$	Relative Maximum

5. We'll first need to find the critical points of the function and determine which ones (if any) fall in the region.

$$f_x = 12 - y + \frac{1}{x^2} \qquad\qquad f_y = -x + \frac{1}{8\left(y-12\right)^2}$$

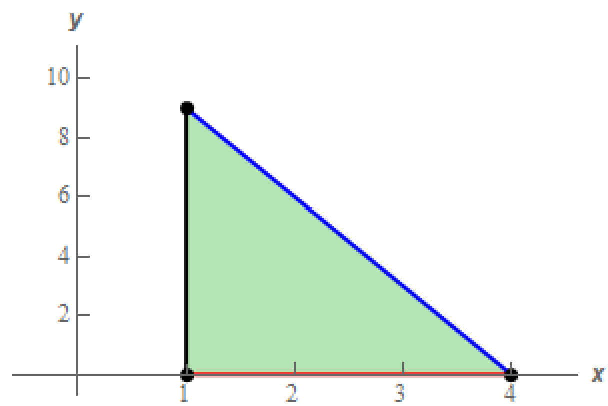
$$12 - y + \frac{1}{x^2} = 0 \qquad \Rightarrow \qquad y - 12 = \frac{1}{x^2}$$

$$\frac{8}{(y-12)^2} - x = 0 \qquad \Rightarrow \qquad \frac{1}{8\left(\frac{1}{x^2}\right)^2} - x = \frac{1}{8}x^4 - x = x\left(\frac{1}{8}x^3 - 1\right) = 0$$

From this it looks like we have $x = 0$ and $x = 2$ however, note that if we use $x = 0$ we get division by zero in the function and so we can't use that. We therefore have only $x = 2$ which gives a single critical point of $\left(2, \frac{49}{4}\right)$. This is outside of the region where working with (see figure/equation below to convince yourself of this) and so we don't need to worry about it either.

Now we need to deal with the boundaries of the region. The region is shown to the right and the equations defining the boundaries are,

$$\begin{aligned} &\text{Top} &&: \ y = 0, \ \ 1 \le x \le 4 \\ &\text{Left} &&: \ x = 1, \ \ 0 \le y \le 9 \\ &\text{Right} &&: \ y = 12 - 3x, \ 1 \le x \le 4 \end{aligned}$$



We now need to run through each of these.

Top : $g(x) = f(x,0) = 12x - \frac{1}{x} + \frac{1}{96} \qquad g'(x) = 12 + \frac{1}{x^2} = \frac{12x^2+1}{x^2}$

We can't have $x = 0$ for the critical point (division by zero in original function) and the derivative is never zero and so there are no critical points and all we need to worry about are the end points. The function evaluations are then,

$$g(1) = f(1,0) = \tfrac{1057}{96} = 11.0104 \qquad g(4) = f(4,0) = \tfrac{4585}{96} = 47.7604$$

Left : $h(y) = f(1,y) = 12 - y - 1 - \frac{1}{8(y-12)} \qquad h'(y) = -1 + \frac{1}{8(y-12)^2} = \frac{-8y^2 + 192y - 1151}{8(y-2)^2}$

We can't have $y = 2$ as a critical point (because the function doesn't exist there) and the derivative is zero at $y = \frac{48 \pm \sqrt{2}}{4} = 11.6464, \ \ 12.3536$ and these are both outside of the range of *y*'s we are working on and so can be ignored. So, again, all we need to worry about are the end points. The function evaluations are then,

$$h(0) = f(1,0) = \tfrac{1057}{96} = 11.0104 \qquad h(9) = f(1,9) = \tfrac{49}{24} = 2.0417$$

Right : $g(x) = f(x, 12-3x) = -\frac{23}{24x} + 3x^2 \qquad g'(x) = \frac{23}{24x^2} + 6x = \frac{23 + 144x^3}{24x^2}$

The only critical point here is $x = \sqrt[3]{\frac{144}{23}} = 1.8431$ (which is in the range of x's we are working on). So we'll need the function at this point and the endpoints. Note however that we've really already done them in the previous two boundaries and so won't list them here.

$$g(1.8431) = f(1.8431, 6.4707) = 9.6711$$

So, from this work it appears that the maximum value is 47.7604 and it occurs at (4, 0) while the minimum value is 2.0417 and it occurs at (1, 9).

6. From the constraint we have $-\frac{3}{\sqrt{2}} \le x \le \frac{3}{\sqrt{2}}$ and $-3 \le y \le 3$ so we are on a closed, bonded region. Therefore, we know that absolute extrema will exist.

$$\begin{aligned} 16xy &= 4x\lambda \qquad \Rightarrow \qquad 4x(4y-\lambda) = 0 \qquad \Rightarrow \qquad x = 0,\ y = \tfrac{1}{4}\lambda \\ 8x^2 &= 2y\lambda \\ 2x^2 + y^2 &= 9 \end{aligned}$$

From the first equation we get two possibilities so let's go through each.

$x = 0$: In this case we can go straight to the constraint to get,

$$y^2 = 9 \qquad \Rightarrow \qquad y = \pm 3 \qquad \Rightarrow \qquad (0,3),\ (0,-3)$$

$y = \frac{1}{4}\lambda$: Here we'll need to plug this into the second equation to get,

$$8x^2 = \tfrac{1}{2}\lambda^2 \qquad \Rightarrow \qquad x^2 = \tfrac{1}{16}\lambda^2$$

Plugging both of these into the constraint gives,

$$2\left(\tfrac{1}{16}\lambda^2\right) + \tfrac{1}{16}\lambda^2 = \tfrac{3}{16}\lambda^2 = 9 \qquad \Rightarrow \qquad \lambda^2 = 48 \qquad \Rightarrow \qquad \lambda = \pm 4\sqrt{3}$$

Each of these values of λ give the following points.

$$\begin{aligned} \lambda = 4\sqrt{3}: &\qquad x = \pm\sqrt{3},\ \ y = \sqrt{3} \qquad \Rightarrow \qquad \left(\sqrt{3},\sqrt{3}\right),\ \left(-\sqrt{3},\sqrt{3}\right) \\ \lambda = -4\sqrt{3}: &\qquad x = \pm\sqrt{3},\ \ y = -\sqrt{3} \qquad \Rightarrow \qquad \left(\sqrt{3},-\sqrt{3}\right),\ \left(-\sqrt{3},-\sqrt{3}\right) \end{aligned}$$

So, we have a total of 6 points to check. Here is the function evaluation for each.

$$f\left(\pm\sqrt{3},\sqrt{3}\right) = 24\sqrt{3} \qquad f\left(\pm\sqrt{3},-\sqrt{3}\right) = -24\sqrt{3} \qquad f(0,\pm 3) = 0$$

The maximum is then $24\sqrt{3}$ which occurs at two points, $\left(\pm\sqrt{3},\sqrt{3}\right)$ while the minimum is $-24\sqrt{3}$ which also occurs at two points, $\left(\pm\sqrt{3},-\sqrt{3}\right)$.

7. For this problem we need to assume that $z \le 0$ in order to make sure that solutions will in fact exist. If we allowed any *z* then we could take *z* as large and positive as we wanted and with sufficiently large *x* and/or *y* we'd be able to meet the constraint. This however would allow the function to grow as large as we wanted. By restricting $z \le 0$ we force all three terms in the constraint to be positive or zero and because the sum of the three positive (or zero) terms must be 16 then neither can be too large and so we know that we will have a minimum and (more importantly) a maximum value of the function.

$$
\begin{aligned}
yz &= 18x\lambda & &\Rightarrow & xyz &= 18x^2\lambda \\
xz &= 2y\lambda & &\Rightarrow & xyz &= 2y^2\lambda \\
xy &= -3\lambda & &\Rightarrow & xyz &= -3z\lambda \\
9x^2 + y^2 - 3z &= 16
\end{aligned}
$$

We got the second set of equations by multiplying the first by *x*, the second by *y* and the third by *z*. If we set the first and second equal as well as the first and third equal we get,

$$
\begin{aligned}
2\lambda\left(9x^2 - y^2\right) &= 0 & &\Rightarrow & \lambda &= 0 \quad \text{or} \quad y^2 = 9x^2 \\
3\lambda\left(6x^2 + z\right) &= 0 & &\Rightarrow & \lambda &= 0 \quad \text{or} \quad z = -6x^2
\end{aligned}
$$

Let's start off by assuming that $\lambda = 0$. In this case the three original equations become,

$$
\begin{aligned}
yz &= 0 & &\Rightarrow & y &= 0 \quad \text{or} \quad z = 0 \\
xz &= 0 & &\Rightarrow & x &= 0 \quad \text{or} \quad z = 0 \\
xy &= 0 & &\Rightarrow & x &= 0 \quad \text{or} \quad y = 0
\end{aligned}
$$

We can't have all three zero since that won't satisfy the constraint. However, notice that if *y*=0 then we must have either *x* or *z* be zero in order to satisfy the second equation. Likewise if *x*=0 then either *y* or *z* must be zero to satisfy the first equation. Finally, if *z*=0 then either *x* or *y* must be zero in order to satisfy the third equation. So, we can't have all three be zero and we can't have only one be zero. However, in this case, we can have two of them be zero. So, if we assume that two are zero and plug these into the constraint to solve for the third we get the following five points,

$$\left(0,0,-\tfrac{16}{3}\right) \qquad \left(0,\pm 4,0\right) \qquad \left(\pm\tfrac{4}{3},0,0\right)$$

Now, let's assume that $\lambda \ne 0$. This forces $y^2 = 9x^2$ and $z = -6x^2$. Plugging these into the constraint gives,

$$9x^2 + 9x^2 + 18x^2 = 36x^2 = 16 \qquad \Rightarrow \qquad x = \pm\tfrac{2}{3}$$

In either case for *x* we get that $y = \pm 2$ and $z = -\tfrac{8}{3}$ and so we get the following set of points.

$$\left(\tfrac{2}{3},2,-\tfrac{8}{3}\right) \qquad \left(\tfrac{2}{3},-2,-\tfrac{8}{3}\right) \qquad \left(-\tfrac{2}{3},2,-\tfrac{8}{3}\right) \qquad \left(-\tfrac{2}{3},-2,-\tfrac{8}{3}\right)$$

Finally all we need to do is plug into the function.

$$f\left(0,0,-\tfrac{16}{3}\right) = f\left(0,\pm 4,0\right) = f\left(\pm\tfrac{4}{3},0,0\right) = 0$$
$$f\left(\tfrac{2}{3},2,-\tfrac{8}{3}\right) = f\left(-\tfrac{2}{3},-2,-\tfrac{8}{3}\right) = -\tfrac{32}{9}$$
$$f\left(-\tfrac{2}{3},2,-\tfrac{8}{3}\right) = f\left(\tfrac{2}{3},-2,-\tfrac{8}{3}\right) = \tfrac{32}{9}$$

The maximum value is then $\frac{32}{9}$ which occurs at $\left(-\frac{2}{3},2,-\frac{8}{3}\right)$ and $\left(\frac{2}{3},-2,-\frac{8}{3}\right)$. The minimum value is $-\frac{32}{9}$ which occurs at $\left(\frac{2}{3},2,-\frac{8}{3}\right)$ and $\left(-\frac{2}{3},-2,-\frac{8}{3}\right)$.