1. (2 pts) Here's the parameterization of the curve and note that it won't be the "standard" one because it is going in the clockwise direction. We'll also need the derivative.

$$\vec{r}\left(t\right)=\left\langle 2\cos t,\sqrt{7}\sin t\right\rangle,\ -\tfrac{\pi}{2}\le t\le\tfrac{\pi}{2} \qquad \vec{r}'\left(t\right)=\left\langle -2\sin t,\sqrt{7}\cos t\right\rangle$$

Next, we'll need the dot product,

$$\begin{aligned}\vec{F}\left(\vec{r}\left(t\right)\right)\bullet\vec{r}'\left(t\right)&=\left\langle 4\cos^2 t,2\cos t-\sqrt{7}\sin t\right\rangle\bullet\left\langle -2\sin t,\sqrt{7}\cos t\right\rangle\\&=-8\sin t\cos^2 t+2\sqrt{7}\cos^2 t-7\sin t\cos t\end{aligned}$$

The integral is then,

$$\begin{aligned}\int_C \vec{F}\bullet d\vec{r}&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}-8\sin t\cos^2 t+2\sqrt{7}\cos^2 t-7\sin t\cos t\,dt\\&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}-8\sin t\cos^2 t+\sqrt{7}\left(1+\cos\left(2t\right)\right)-\tfrac{7}{2}\sin\left(2t\right)dt\\&=\left(-\tfrac{8}{3}\cos^3 t+\sqrt{7}\left(t+\tfrac{1}{2}\sin\left(2t\right)\right)+\tfrac{7}{4}\cos\left(2t\right)\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\boxed{\sqrt{7}\,\pi}\end{aligned}$$

3. (2 pts) Not much here other than to use the Fundamental Theorem of Calculus.

$$\int_C \nabla f\bullet d\vec{r}=f\left(\vec{r}\left(3\right)\right)-f\left(\vec{r}\left(1\right)\right)=f\left(0,\tfrac{3}{2},7\right)-f\left(0,\tfrac{1}{2},9\right)=\left(-\tfrac{52}{3}\right)-\left(-80\right)=\boxed{\tfrac{188}{3}}$$

8. (3 pts) A quick check to verify that it is conservative to make sure there isn't a typo here.

$$P=4x^2\mathbf{e}^{4x-2y}+2x\mathbf{e}^{4x-2y}-4\mathbf{e}^{4x-2y} \qquad Q=2\mathbf{e}^{4x-2y}-2x^2\mathbf{e}^{4x-2y}-2y$$

$$P_y=-8x^2\mathbf{e}^{4x-2y}-4x\mathbf{e}^{4x-2y}+8\mathbf{e}^{4x-2y} \qquad Q_x=8\mathbf{e}^{4x-2y}-4x\mathbf{e}^{4x-2y}-8x^2\mathbf{e}^{4x-2y}$$

It is conservative so in this case we'll integrate *Q* with respect to *y* first.

$$f\left(x,y\right)=\int Q\,dy=\int 2\mathbf{e}^{4x-2y}-2x^2\mathbf{e}^{4x-2y}-2y\,dy=-\mathbf{e}^{4x-2y}+x^2\mathbf{e}^{4x-2y}-y^2+g\left(x\right)$$

Differentiate with respect to *x* and set equal to *P*.

$$f_x=-4\mathbf{e}^{4x-2y}+2x\mathbf{e}^{4x-2y}+4x^2\mathbf{e}^{4x-2y}+g'\left(x\right)=4x^2\mathbf{e}^{4x-2y}+2x\mathbf{e}^{4x-2y}-4\mathbf{e}^{4x-2y}$$

$$g'\left(x\right)=0 \qquad \Rightarrow \qquad g\left(x\right)=c$$

The potential function is then,

$$f\left(x,y\right)=\left(x^2-1\right)\mathbf{e}^{4x-2y}-y^2+c$$

In order to evaluate the integral we need the initial and final point of *C* and because we know the center and radius of the circle we can get these easily enough : Initial Point : (2, -2), Final Point : (2, 8). The integral is then (ignoring the *c* because it will cancel),

$$\int_C \vec{F}\bullet d\vec{r}=f\left(2,8\right)-f\left(2,-2\right)=3\mathbf{e}^{-8}-64-\left(3\mathbf{e}^{12}-4\right)=\boxed{3\mathbf{e}^{-8}-3\mathbf{e}^{12}-60}$$

10. (3 pts)

(a) Here's a parameterization of the curve.

$$\vec{r}\left(t\right)=\left\langle 2\cos t,2\sin t\right\rangle,\ \ 0\le t\le 2\pi \qquad\qquad \vec{r}'\left(t\right)=\left\langle -2\sin t,2\cos t\right\rangle$$

The integral is then (be careful about the *P* and the *Q*!),

$$\oint_C \left(4x+y\right)dy-\left(8y-3\right)dx=\int_0^{2\pi}\left(8\cos t+2\sin t\right)\left(2\cos t\right)-\left(16\sin t-3\right)\left(-2\sin t\right)dt$$

$$=\int_0^{2\pi}16\cos^2 t+4\sin t\cos t+32\sin^2 t-6\sin t\,dt$$

$$=\int_0^{2\pi}16+4\sin t\cos t+16\sin^2 t-6\sin t\,dt$$

$$=\int_0^{2\pi}16+2\sin\left(2t\right)+8\left(1-\cos\left(2t\right)\right)-6\sin t\,dt=\boxed{48\pi}$$

(b) Here's the integral after applying Green's Theorem.

$$\oint_C \left(4x+y\right)dy-\left(8y-3\right)dx=\iint_D 4-\left(-8\right)dA=12\iint_D dA=12\left(4\pi\right)=\boxed{48\pi}$$

Note that we didn't need to actually do the integral since it's just the area of the disk. Also note that we did get the same answer (as we should…).

Not Graded

2. We'll need the derivative of the parameterization and the dot product.

$$\vec{r}'\left(t\right)=2t\,\vec{i}-2\,\vec{j}+2t\,\vec{k}$$

$$\vec{F}\left(\vec{r}\left(t\right)\right)\bullet\vec{r}\left(t\right)=\left(\cos\left(t^2\right)\vec{i}-\left(1+t^2\right)^2\left(-2t\right)^3\vec{j}+\left(4t^2+1+t^2\right)\vec{k}\right)\bullet\left(2t\,\vec{i}-2\vec{j}+2t\,\vec{k}\right)$$

$$=2t\cos\left(t^2\right)-16t^3\left(1+2t^2+t^4\right)+2t\left(5t^2+1\right)=2t\cos\left(t^2\right)-16t^7-32t^5-6t^3+2t$$

The integral is then,

$$\int_C \vec{F}\bullet d\vec{r}=\int_{-1}^{0}2t\cos\left(t^2\right)-16t^7-32t^5-6t^3+2t\,dt=\boxed{\tfrac{47}{6}-\sin\left(1\right)=6.99186}$$

4. This is not as difficult as it might seem at first glance. First we need to go back and redo the integral from **1** only this time we need to do is on the interval $0\le t\le 2\pi$, *i.e.* one complete revolution.

$$\int_C \vec{F}\bullet d\vec{r}=\int_0^{2\pi}-8\sin t\cos^2 t+2\sqrt{7}\cos^2 t-7\sin t\cos t\,dt=\boxed{2\sqrt{7}\,\pi}$$

So, we've got a line integral on a closed path that is not zero and so there is no way that this integral can be independent of path.

5. First run through the test.

$$P = 6x^4 - 8y + x^2y^2 \qquad P_y = -8 + 2x^2y$$
$$Q = 8x + x^2y^2 \qquad Q_x = 8 + 2xy^2$$

So, $P_y \neq Q_x$ and so the vector field is **NOT** conservative.

6. First run through the test.

$$P = y^4 + 18x^2y^2 - 2xy + 6 \qquad P_y = 4y^3 + 36x^2y - 2x$$
$$Q = -\left(x^2 + 24y - 12x^3y - 4xy^3\right) \qquad Q_x = -2x + 36x^2y + 4y^3$$

So, the vector field is conservative. We'll start off by integrating *P* with respect to *x* first.

$$f(x,y) = \int P\,dx = \int y^4 + 18x^2y^2 - 2xy + 6\,dx = xy^4 + 6x^3y^2 - x^2y + 6x + g(y)$$

Differentiate with respect to *y* and set equal to *Q*.

$$f_y = 4xy^3 + 12x^3y - x^2 + g'(y) = 12x^3y + 4xy^3 - x^2 - 24y$$
$$\Rightarrow\ g'(y) = -24y \ \Rightarrow\ g(y) = -12y^2 + c$$

The potential function is then : $\boxed{f(x,y) = 6x^3y^2 + xy^4 - x^2y - 12y^2 + 6x + c}$

7. We already found the potential function for this in **6** and so all we need to do is compute the integral (ignoring the *c* since it will cancel out).

$$\int_C \vec{F} \bullet d\vec{r} = f(3,0) - f(1,-2) = 18 - 0 = \boxed{18}$$

9. Note that from the parametric curve is should be obvious that we can assume that both *x* and *z* to be positive. Next, because we don't know how to verify this is conservative (yet) we'll have to believe that it is. Here are the various parts of the vector field.

$$P = 4 + \frac{y}{x} - 2xz^3 \qquad Q = \ln(xz) \qquad R = \frac{y}{z} - 3x^2z^2$$

Integrate with respect to *x* first

$$f(x,y,z) = \int 4 + \frac{y}{x} - 2xz^3\,dx = 4x + y\ln x - x^2z^3 + g(y,z)$$

Differentiating with respect to *y* and setting equal to *Q* gives,

$$f_y = \ln x + g_y(y,z) = \ln(xz) = \ln x + \ln z \qquad \Rightarrow \qquad g_y(y,z) = \ln z$$
$$g(y,z) = y\ln z + h(z)$$

The potential function is now,

$$f(x,y,z) = 4x + y\ln x - x^2z^3 + y\ln z + h(z)$$

Differentiating with respect to *z* and setting equal to *R* gives,

$$f_z = \frac{y}{z} - 3x^2z^2 + h'(z) = \frac{y}{z} - 3x^2z^2 \quad \Rightarrow \quad h'(z) = 0 \quad \Rightarrow \quad h(z) = c$$

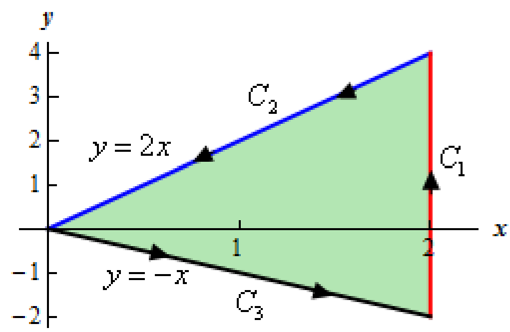
The potential function is then,

$$f(x,y,z) = 4x + y\ln x - x^2z^3 + y\ln z + c = 4x + y\ln(xz) - x^2z^3 + c$$

Finally, the integral (ignoring the *c* because it will cancel),

$$\int_C \vec{F} \bullet d\vec{r} = f(\vec{r}(2)) - f(r(0)) = f(2,4,5) - f(2,-2,1) = \boxed{496 - 2\ln 2 - 4\ln 10 = 485.403}$$

11. (a) A sketch of the curve *C* and the enclosed region *D* is to the right. Here are the parameterizations for each curve and because we need to pay attention to direction we will not be able to use the equation for line C_2.



$$C_1 : \vec{r}(t) = \langle 2, t \rangle,\ -2 \le t \le 4$$
$$C_2 : \vec{r}(t) = \langle 2-2t, 4-4t \rangle,\ 0 \le t \le 1$$
$$C_3 : \vec{r}(t) = \langle t, -t \rangle,\ 0 \le t \le 2$$

Here's the integral done directly for each curve. Note the second integral was supposed to be *dy*!

$$\oint_{C_1} 4xy\,dx + (x^2+2y)\,dy = \int_{-2}^{4} 4(2)(t)(0) + (2^2 + 2(t))(1)\,dt = \int_{-2}^{4} 4 + 2t\,dt = \underline{36}$$

$$\oint_{C_2} 4xy\,dx + (x^2+2y)\,dy = \int_0^1 4(2-2t)(4-4t)(-2) + \left[(2-2t)^2 + 2(4-4t)\right](-4)\,dt$$

$$= \int_0^1 -112 + 192t - 80t^2\,dt = \underline{-\tfrac{128}{3}}$$

$$\oint_{C_3} 4xy\,dx + (x^2+2y)\,dy = \int_0^2 4(t)(-t)(1) + \left[t^2 + 2(-t)\right](-1)\,dt = \int_0^2 2t - 5t^2\,dt = \underline{-\tfrac{28}{3}}$$

The integral is then,

$$\oint_C 4xy\,dx + (x^2+2y)\,dy = 36 - \tfrac{128}{3} - \tfrac{28}{3} = \boxed{-16}$$

(b) Using Green's Theorem the integral is,

$$\oint_C 4xy\,dx + (x^2+2y)\,dy = \iint_D 2x - (4x)\,dA = \iint_D -2x\,dA$$

We'll use the following for limits for *D* : $0 \le x \le 2,\ -x \le y \le 2x$. The integral is then,

$$\oint_C 4xy\,dx + \left(x^2 + 2y\right)dy = \int_0^2 \int_{-x}^{2x} -2x\,dy\,dx = \int_0^2 -6x^2\,dx = \boxed{-16}$$