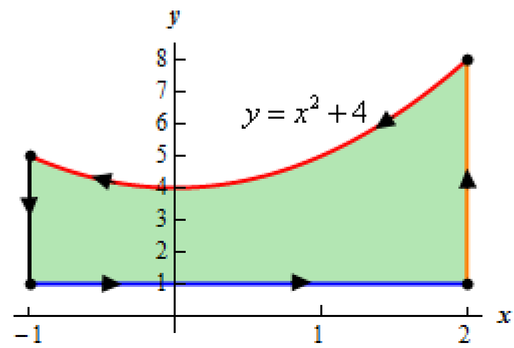
Math 2415 **Homework Set 10 – Solutions** **10 Points**

1. A sketch of the curve/region is to the right. The limits that we'll use for the double integral are :

$-1 \le x \le 2,\ 1 \le y \le x^2 + 4$.



The integral is,

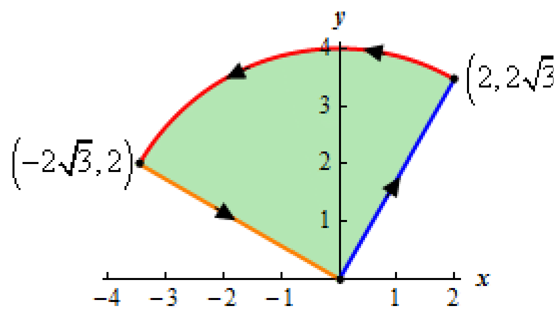
$$\oint_C 16xy\,dx + \left(1 - x^2\right)dy = \iint_D -2x - 16x\,dA$$

$$= \int_{-1}^{2}\int_{1}^{x^2+4} -18x\,dy\,dx$$

$$= \int_{-1}^{2} -27x^2 - \tfrac{9}{2}x^4\,dx = \boxed{-\tfrac{297}{2}}$$

2. A sketch of the curve/region is to the right. The limits are,

$$0 \le r \le 4 \qquad \tfrac{\pi}{3} = \tan^{-1}\left(\sqrt{3}\right) \le \theta \le \tan^{-1}\left(\tfrac{-1}{\sqrt{3}}\right) + \pi = \tfrac{5\pi}{6}$$



The integral is then,

$$\int_C \vec{F}\bullet d\vec{r} = \iint_D -3x^2 - \left(3y^2 + 2y\right)dA$$

$$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{6}}\int_{0}^{4} -3r^3 - 2r^2\sin\theta\,dr\,d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} -192 - \tfrac{128}{3}\sin\theta\,d\theta = \boxed{-\tfrac{32}{3}\left(2 + 2\sqrt{3} + 9\pi\right)}$$

3.

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \frac{y^2}{xz} & x\ln(x) & 4z + x^2 - y^3 \end{vmatrix}$$

$$= -3y^2\vec{i} - \tfrac{y^2}{xz^2}\vec{j} + \left(1 + \ln(x)\right)\vec{k} - \tfrac{2y}{xz}\vec{k} - 2x\vec{j} - (0)\vec{i}$$

$$= \boxed{-3y^2\vec{i} - \left(\tfrac{y^2}{xz^2} + 2x\right)\vec{j} + \left(1 + \ln(x) - \tfrac{2y}{xz}\right)\vec{k}}$$

$$\operatorname{div}\vec{F} = -\tfrac{y^2}{x^2 z} + 0 + 4 = \boxed{4 - \tfrac{y^2}{x^2 z}}$$

4. We computed the curl of this in **3** and we saw there that it was not zero and so this vector field is not conservative.

5. Here's the curl for his vector field

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4+\frac{y}{x}-2xz^3 & \ln(xz) & \frac{y}{z}-3x^2z^2 \end{vmatrix}$$

$$= \tfrac{1}{z}\vec{i} - 6xz^2\,\vec{j} + \tfrac{1}{x}\vec{k} - \tfrac{1}{x}\vec{k} + 6xz^2\,\vec{j} - \tfrac{1}{z}\vec{i} = \boxed{\vec{0}}$$

So, this vector field is conservative, but then we already knew that because we found the potential function for this vector field in the previous homework set.

6. Let's call the points,

$$P = (1,-1,0) \qquad Q = (2,0,3) \qquad R = (1,4,1)$$

The following two vectors will live in the plane.

$$\overrightarrow{PQ} = \langle 1,1,3\rangle \qquad \overrightarrow{PR} = \langle 0,5,1\rangle$$

The cross product of these two will be normal to the plane.

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 3 \\ 0 & 5 & 1 \end{vmatrix} = \vec{i} + 5\vec{k} - \vec{j} - 15\vec{i} = -14\vec{i} - \vec{j} + 5\vec{k}$$

Using the point *P* the equation of the plane is,

$$-14(x-1)-(y+1)+5z=0 \quad\Rightarrow\quad -14x-y+5z=-13 \quad\Rightarrow\quad z=\tfrac{1}{5}(-13+14x+y)$$

The parameterization of the plane can then be written as,

$$\vec{r}(x,y) = \left\langle x, y, \tfrac{1}{5}(-13+14x+y)\right\rangle$$

There will be no restrictions on *x* and *y* since we did not have any restrictions on the plane. Note as well that we could just have easily solved the equation of the plane for *x* or *y* and gotten two different (and potentially easier to deal with depending upon the problem) parameterizations.

7. The parameterization is : $\vec{r}(x,y) = \left\langle x, y, 8x^2+8y^2+3\right\rangle$. The limits on the variables come from the intersection.

$$9 = 8x^2+8y^2+3 \qquad\Rightarrow\qquad x^2+y^2=\tfrac{3}{4}$$

So, (*x*, *y*) come from the disk : $x^2+y^2 \le \frac{3}{4}$

8. Using "polar" conversion formulas the parameterization is : $r(y,\theta)=\left\langle \sqrt{3}\cos\theta, y, \sqrt{3}\sin\theta\right\rangle$. The limits are : $0\le\theta\le 2\pi,\ -2\le y\le 15$.

9. Using the spherical conversion formulas the parameterization is :

$$\vec{r}(\theta,\varphi)=\langle 4\sin\varphi\cos\theta, 4\sin\varphi\sin\theta, 4\cos\varphi\rangle$$

The limits are : $0\le\theta\le 2\pi,\ \frac{\pi}{2}\le\varphi\le\pi$. Note the limits on φ are because we have the lower half.

10. First find $\vec{r}_u$ and $\vec{r}_v$.

$$\vec{r}(u,v)=(7u-uv)\vec{i}+(1-u^2)\vec{j}+(v^2-4v)\vec{k}$$

$$\vec{r}_u(u,v)=(7-v)\vec{i}-2u\,\vec{j} \qquad \vec{r}_v(u,v)=-u\,\vec{i}+(2v-4)\vec{k}$$

Now, find the cross product of the two

$$\vec{r}_u\times\vec{r}_v=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 7-v & -2u & 0\\ -u & 0 & 2v-4\end{vmatrix}=-2u(2v-4)\vec{i}-(7-v)(2v-4)\vec{j}-2u^2\,\vec{k}$$

Now, find the values of *u* and *v* that give the point (-2, -3, 12).

$$\begin{aligned}-2&=7u-uv &&\Rightarrow && u=-2,\quad v=6\\ -3&=1-u^2 &&\Rightarrow && u=\pm 2\\ 12&=v^2-4v &&\Rightarrow && v=6,-2\end{aligned}$$

From the last two we get all possible values of *u* and *v* and then we get the actual values that we need from the first equation. Now, evaluate $\vec{r}_u\times\vec{r}_v$ at $u=-2$ and $v=6$,

$$\vec{r}_u\times\vec{r}_v=32\vec{i}-8\,\vec{j}-8\,\vec{k}$$

The equation of the tangent plane is then,

$$32(x+2)-8(y+3)-8(z-12)=0$$

$$32x-8y-8z=-136 \qquad\to\qquad z=17+4x-y$$

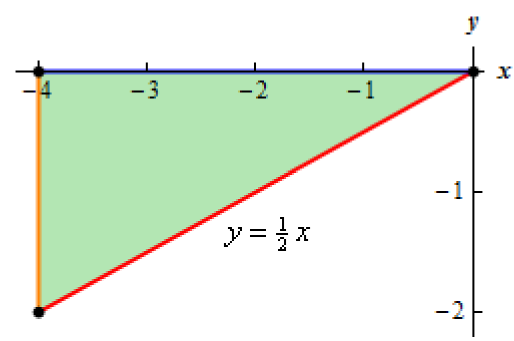
The parameterization is then : $\vec{r}(x,y)=\langle x, y, 17+4x-y\rangle$.

11. A sketch of the region is to the right. The parameterization of the surface is,

$$\vec{r}(x,y)=\langle x, y, 1+x^2+8y\rangle$$

$$\vec{r}_x=\langle 1,0,2x\rangle \qquad \vec{r}_y=\langle 0,1,8\rangle$$

The cross product is,



$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x \\ 0 & 1 & 8 \end{vmatrix} = -2x\,\vec{i} - 8\,\vec{j} + \vec{k}$$

The magnitude of the cross product is,

$$\| \vec{r}_x \times \vec{r}_y \| = \sqrt{4x^2 + 64 + 1} = \sqrt{4x^2 + 65}$$

So, it looks like we'll need the following limits on *D* : $-4 \le x \le 0,\ \frac{1}{2}x \le y \le 0$. The surface are is then,

$$S = \iint\limits_D \sqrt{4x^2 + 65}\, dA = \int_{-4}^{0} \int_{\frac{1}{2}x}^{0} \sqrt{4x^2 + 65}\, dy\, dx = \int_{-4}^{0} -\tfrac{1}{2} x \sqrt{4x^2 + 65}\, dx = \boxed{-\tfrac{1}{24}\left(65^{\frac{3}{2}} - 129^{\frac{3}{2}}\right)}$$

12. We already have the parameterization for this one so here is everything we need.

$$\vec{r}_u = -9v\,\vec{j} - \vec{k} \qquad \vec{r}_v = \vec{i} - 9u\vec{j}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -9v & -1 \\ 1 & -9u & 0 \end{vmatrix} = -9u\vec{i} - \vec{j} + 9v\,\vec{k} \qquad \| \vec{r}_u \times \vec{r}_v \| = \sqrt{81u^2 + 1 + 81v^2}$$

It looks like we'll be needing polar coordinates here so the limits are,

$$0 \le \theta \le 2\pi \qquad 0 \le r \le 2$$

The surface area is then,

$$S = \iint\limits_D \sqrt{1 + 81u^2 + 81v^2}\, dA = \int_0^{2\pi} \int_0^2 r\sqrt{1 + 81r^2}\, dr\, d\theta = \int_0^{2\pi} \tfrac{1}{243}\left(324^{\frac{3}{2}} - 1\right) d\theta = \boxed{\tfrac{2\pi}{243}\left(324^{\frac{3}{2}} - 1\right)}$$