1. (3 pts) First $z = 6 - 3x - y$ and so the integral is,
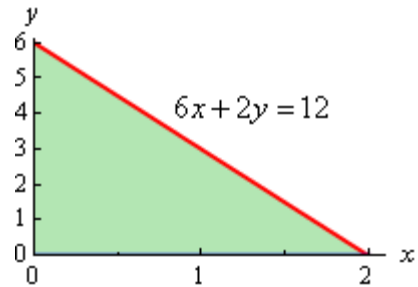
$$\iint_S z + 3x \, dS = \iint_D 6 - 3x - y + 3x \sqrt{(-3)^2 + (-1)^2 + 1} \, dA$$

$$= \sqrt{11} \iint_D 6 - y \, dA$$

The region *D* is sketched to the right and the limits will be,

$$0 \le x \le 2, \quad 0 \le y \le 6 - 3x$$

$6x + 2y = 12$

The integral is then,

$$\iint_S z + 3x \, dS = \sqrt{11} \int_0^2 \int_0^{6-3x} 6 - y \, dy \, dx = \sqrt{11} \int_0^2 18 - \tfrac{9}{2} x^2 \, dx = \boxed{24\sqrt{11}}$$

3. (4 pts) A sketch of *S* is to the right. Note as well that the orientation of the axes are not standard. This is being looked at "top down" so to speak to see the plane on the right side of the object.

We now need parameterizations for each.

S_1 : Cylinder

$$\vec{r}(y, \theta) = 3\cos\theta\, \vec{i} + y\, \vec{j} + 3\sin\theta\, \vec{k}$$

$$0 \le \theta \le 2\pi, \, -1 \le y \le x + 6 = 3\cos\theta + 6$$

$S_3 : y = -1$

$S_2 : y = x + 6$

$S_1 : x^2 + z^2 = 9$

S_2 : Front of cylinder

$\qquad y = x + 6, (x, z)$ is in the disk of radius 3 centered at origin (in polar of course)

S_3 : Back of cylinder

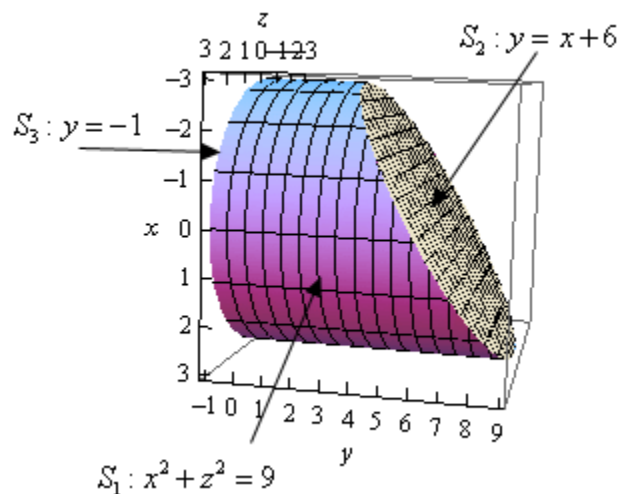$\qquad y = -1, (x, z)$ is in the disk of radius 3 centered at origin

Now the integrals for each of these surfaces.

S_1 : Cylinder

In this case we'll need to do the cross product stuff so let's get that taken care of first.

$$\vec{r}_y = \vec{j} \qquad\qquad \vec{r}_\theta = -3\sin\theta\, \vec{i} + 3\cos\theta\, \vec{k}$$

$$\vec{r}_y \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ -3\sin\theta & 0 & 3\cos\theta \end{vmatrix} = 3\cos\theta\, \vec{i} + 3\sin\theta\, \vec{k} \qquad\qquad \| \vec{r}_y \times \vec{r}_\theta \| = \sqrt{9\cos^2\theta + 9\sin^2\theta} = 3$$

Now the integral.

$$\iint_{S_1} x - 2y\,dS = \iint_D (3\cos\theta - 2y)(3)\,dA = \int_0^{2\pi}\int_{-1}^{3\cos\theta+6} 9\cos\theta - 6y\,dy\,d\theta$$

$$= \int_0^{2\pi} -105 - 45\cos(\theta)\,d\theta = \boxed{-210\pi}$$

S_2 : Front of cylinder

$$\iint_{S_2} x - 2y\,dS = \iint_D (x - 2(x+6))\sqrt{1+1+0}\,dA = -\sqrt{2}\iint_D x + 12\,dA$$

$$= -\sqrt{2}\int_0^{2\pi}\int_0^3 r^2\cos\theta + 12r\,dr\,d\theta = -\sqrt{2}\int_0^{2\pi} 9\cos(\theta) + 54\,d\theta = \boxed{-108\sqrt{2}\,\pi}$$

S_3 : Back of cylinder

$$\iint_{S_3} x - 2y\,dS = \iint_D (x+2)\sqrt{0+1+0}\,dA = \int_0^{2\pi}\int_0^3 r^2\cos\theta + 2r\,dr\,d\theta = \int_0^{2\pi} 9\cos(\theta) + 9\,d\theta = \boxed{18\pi}$$

The overall integral is then,

$$\iint_S x - 2y\,dS = -210\pi - 108\sqrt{2}\,\pi + 18\pi = \boxed{-192\pi - 108\sqrt{2}\,\pi}$$

4. (3 pts) We'll first need the gradient.

$$f(x,y,z) = x + y^2 + z^2 - 4 \qquad\qquad \nabla f = \langle 1, 2y, 2z \rangle$$

Notice that this is oriented in the positive *x* direction (because the *x* component is positive) so we'll need to use the negative of this.

$$-\nabla f = \langle -1, -2y, -2z \rangle$$

The region *D* comes from the intersection of the two surfaces.

$$3 = 4 - y^2 - z^2 \qquad\qquad y^2 + z^2 = 1$$

So, *D* is a circle of radius 1 centered at the origin so we'll need the polar coordinates,

$$x = r\cos\theta, \quad z = r\sin\theta$$

 Now the dot product.

$$\vec{F}\cdot\nabla f = \langle 12x, z, 6 - y \rangle \cdot \langle -1, -2y, -2z \rangle = -12x - 2yz - 2z(6 - y)$$

$$= -12(4 - y^2 - z^2) - 12z = -12(z + 4 - y^2 - z^2)$$

Notice that I didn't bother with the $\|\nabla f\|$ since they were just going to cancel when we go to do the integral. Speaking of which,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D -12(z + 4 - y^2 - z^2)\,dA = -12\int_0^{2\pi}\int_0^1 r^2\sin\theta + 4r - r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} \tfrac{7}{4} + \tfrac{1}{3}\sin\theta\,d\theta = \boxed{-42\pi}$$

Not Graded

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2. The integral is,

$$\iint\limits_{S} 8z - 24\, dS = \iint\limits_{D}\left[8\left(x^2 + y^2 + 3\right) - 24\right]\sqrt{\left(2x\right)^2 + \left(2y\right)^2 + 1}\, dA$$

$$= \iint\limits_{D} 8\left(x^2 + y^2\right)\sqrt{4x^2 + 4y^2 + 1}\, dA$$

The region *D* will be given by the intersection of the two surfaces.

$$4 = x^2 + y^2 + 3 \qquad \Rightarrow \qquad x^2 + y^2 = 1$$

So, it looks like we have a disk and so we'll need to finish with polar coordinates.

$$\iint\limits_{S} 8z - 24\, dS = \iint\limits_{D} 8r^2\sqrt{4r^2 + 1}\, dA = \int_{0}^{2\pi}\int_{0}^{1} 8r^3\sqrt{4r^2 + 1}\, dr\, d\theta$$

For the integral use the substitution : $u = 4r^2 + 1 \quad \rightarrow \quad r^2 = \tfrac{1}{4}\left(u - 1\right)$.

$$\iint\limits_{S} 8z - 24\, dS = \int_{0}^{2\pi}\int_{0}^{1} 8r^3\sqrt{4r^2 + 1}\, dr\, d\theta = \int_{0}^{2\pi}\int_{1}^{5} \tfrac{1}{4}\left(u - 1\right)\sqrt{u}\, dr\, d\theta$$

$$= \tfrac{1}{4}\int_{0}^{2\pi}\int_{1}^{5} u^{\frac{3}{2}} - u^{\frac{1}{2}}\, dr\, d\theta = \int_{0}^{2\pi} \tfrac{1}{15} + \tfrac{5\sqrt{5}}{3}\, d\theta = \boxed{2\pi\left(\tfrac{1}{15} + \tfrac{5\sqrt{5}}{3}\right) = 23.8349}$$

5. We're using the surface from **3** so let's get the info from there copied to here…

S_1 : Cylinder

$$\vec{r}\left(y, \theta\right) = 3\cos\theta\, \vec{i} + y\, \vec{j} + 3\sin\theta\, \vec{k} \qquad\qquad 0 \le \theta \le 2\pi,\, -1 \le y \le x + 6 = 3\cos\theta + 6$$

S_2 : Front of cylinder

$\qquad y = x + 6, \left(x, z\right)$ is in the disk of radius 3 centered at origin (in polar of course)

S_3 : Back of cylinder

$\qquad y = -1, \left(x, z\right)$ is in the disk of radius 3 centered at origin

Now we need to go through each of the integrals.

S_1 : Cylinder

In this case we'll need $\vec{r}_y \times \vec{r}_\theta$ which we found in **3** to be : $\vec{r}_y \times \vec{r}_\theta = 3\cos\theta\, \vec{i} + 3\sin\theta\, \vec{k}$. This will always point outwards and so we have the correct orientation here. The dot product is,

$$\vec{F} \bullet \left(\vec{r}_y \times \vec{r}_\theta \right) = \left\langle z, y-6, -3 \right\rangle \bullet \left\langle 3\cos\theta, 0, 3\sin\theta \right\rangle$$

$$= \left\langle 3\sin\theta, y-6, -3 \right\rangle \bullet \left\langle 3\cos\theta, 0, 3\sin\theta \right\rangle = 9\cos\theta\sin\theta - 9\sin\theta$$

The integral is then,

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_D 9\cos\theta\sin\theta - 9\sin\theta\, dA = \int_0^{2\pi} \int_{-1}^{3\cos\theta+6} 9\cos\theta\sin\theta - 9\sin\theta\, dy\, d\theta$$

$$= \int_0^{2\pi} 27\cos^2\theta\sin\theta + 36\cos\theta\sin\theta - 63\sin\theta\, d\theta$$

$$= \left(9\cos^3\theta - 9\cos(2\theta) + 63\cos\theta \right) \Big|_0^{2\pi} = \boxed{0}$$

S_2 : Front of cylinder

The gradient is

$$f(x,y,z) = y - x - 6 \qquad\qquad \nabla f = \left\langle -1, 1, 0 \right\rangle$$

This points in the positive y direction and so is pointing outward. The integral is now.

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_D \left\langle z, x, -3 \right\rangle \cdot \left\langle -1, 1, 0 \right\rangle dA = \iint_D x - z\, dA$$

$$= \int_0^{2\pi} \int_0^1 r^2 \left(\cos\theta - \sin\theta \right) dr\, d\theta = \int_0^{2\pi} \tfrac{1}{3} \left(\cos\theta - \sin\theta \right) d\theta = \boxed{0}$$

S_3 : Back of cylinder

The gradient is

$$f(x,y,z) = y + 1 \qquad\qquad \nabla f = \left\langle 0, 1, 0 \right\rangle \qquad\qquad -\nabla f = \left\langle 0, -1, 0 \right\rangle$$

Since this is the back of the cylinder we need the orientation to be in the negative y direction to be outward. Note that we could just have easily done this directly using $\vec{n} = -\vec{j}$. The integral is

$$\iint_{S_3} \vec{F} \cdot d\vec{S} = \iint_D \left\langle z, -7, -3 \right\rangle \cdot \left\langle 0, -1, 0 \right\rangle dA = \iint_D 7\, dA = 7\left(\text{Area of D} \right) = \boxed{63\pi}$$

The overall integral is then,

$$\iint_S \vec{F} \cdot d\vec{S} = 0 + 0 + 63\pi = \boxed{63\pi}$$

Do not get excited about or come to expect a couple of the integrals being zero. It just worked out to be that way for this problem.

6. In this case we'll use Stokes' Theorem in the following direction.

$$\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

where *C* is the boundary of the surface *S*. So, this will be the circle of radius 2 in the *xz*-plane that lies at

$$y = 2x^2 + 2z^2 = 2\left(x^2 + z^2\right) = 2(4) = 8 \qquad \Rightarrow \qquad y = 8$$

The parameterization of this curve is then,

$$\vec{r}(t) = \langle 2\cos\theta, 8, 2\sin\theta\rangle \qquad \vec{r}'(t) = \langle -2\sin\theta, 0, 2\cos\theta\rangle$$

Now, the dot product is,

$$\vec{F} \cdot \vec{r}' = \langle y, y^2 - z^3, -12x\rangle \cdot \langle -2\sin\theta, 0, 2\cos\theta\rangle = -2\sin\theta(y) + 2\cos\theta(-12x) = -16\sin\theta - 48\cos^2\theta$$

The integral is this

$$\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -16\sin\theta - 48\cos^2\theta\, d\theta = \boxed{-48\pi}$$

7. In this case we'll use Stokes' Theorem in the following direction

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F} \cdot d\vec{S}$$

So, we need the curl of the vector field.

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y & x^3 & -(5z+2) \end{vmatrix} = 3x^2\vec{k} - 3\vec{k} = \left(3x^2 - 3\right)\vec{k}$$

Now, there are a variety of surfaces we could use here but it seems like one of the easiest to use is $z = x^2 + y^2 - 5$ and we'll need it to be oriented downwards (remember that the as we walk along the curve the surface must be on the left and our head will then point in the direction of the normal vector). The gradient is then,

$$f(x, y, z) = x^2 + y^2 - 5 - z \qquad \nabla f = \langle 2x, 2y, -1\rangle$$

This gives the correct orientation and the region D is $x^2 + y^2 = 2$. So, the disk of radius $\sqrt{2}$ centered at the origin. The integral is then,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = \iint_D \langle 0, 0, 3x^2 - 3\rangle \cdot \langle 2x, 2y, -1\rangle\, dA = \iint_D 3 - 3x^2\, dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} 3r - 3r^3 \cos^2\theta\, dr\, d\theta = \int_0^{2\pi} 3 - 3\cos^2\theta\, d\theta = \boxed{3\pi}$$

8. We will be using the Divergence Theorem in the following direction.

$$\iint_S \vec{F} \cdot d\vec{r} = \iiint_E \operatorname{div}\vec{F}\, dV$$

So we will need the divergence and *E*.

$$\operatorname{div}\vec{F} = 4z + 2 - 4 = 4z - 2$$

E is the portion of a sphere so we'll be doing this integral in spherical coordinates and the limits are,

$$0 \le \varphi \le \frac{\pi}{2} \qquad 0 \le \theta \le \frac{\pi}{2} \qquad 2 \le \rho \le 3$$

The integral is then.

$$\iint\limits_{S} \vec{F} \cdot d\vec{r} = \iiint\limits_{E} 4z - 2\, dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_2^3 4\rho^3 \cos\varphi \sin\varphi - 2\rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 65 \cos\varphi \sin\varphi - \tfrac{38}{3} \sin\varphi \, d\theta \, d\varphi$$

$$= \tfrac{\pi}{2} \int_0^{\frac{\pi}{2}} 65 \cos\varphi \sin\varphi - \tfrac{38}{3} \sin\varphi \, d\varphi = \boxed{\dfrac{119\pi}{12}}$$