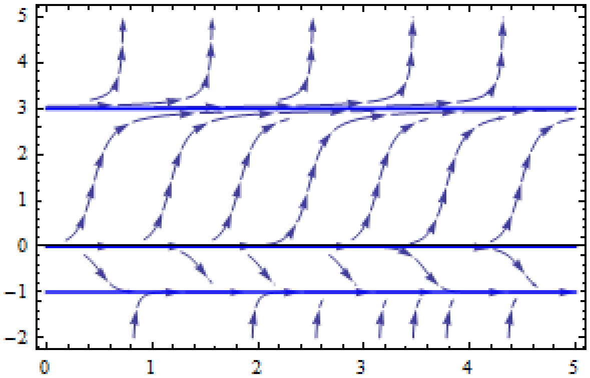
1. (2 pts) The derivative will be zero at $y=-1$, $y=0$ and $y=3$. A sketch of some solutions is to the right (note I didn't include the direction field because of computer issues). From this we can see the long term behavior and dependence on $y(0)$ is,



$y(0)>3$	$y(t)\to\infty$
$0<y(0)\le 3$	$y(t)\to 3$
$y(0)=0$	$y(t)=0$
$y(0)<0$	$y(t)\to -1$

4. (3 pts) Don't forget that the y' needs a coefficient of 1 before we do any of the work.

$$y' + \left(\tfrac{3}{2}-\tfrac{4}{t}\right)y = \tfrac{1}{2}t^5\mathbf{e}^{-\frac{3}{2}t}\sin(\pi t) \qquad \mu = \mathbf{e}^{\int \frac{3}{2}-\frac{4}{t}\,dt} = \mathbf{e}^{\frac{3}{2}t-4\ln t} = t^{-4}\mathbf{e}^{\frac{3}{2}t}$$

$$\int \left(t^{-4}\mathbf{e}^{\frac{3}{2}t}y\right)'\,dt = \int \tfrac{1}{2}t\sin(\pi t)\,dt$$

$$t^{-4}\mathbf{e}^{\frac{3}{2}t}y = \tfrac{1}{2\pi^2}\left(\sin(\pi t)-t\pi\cos(\pi t)\right)+c \qquad \Rightarrow \qquad \underline{y(t)=\tfrac{1}{2\pi^2}t^4\mathbf{e}^{-\frac{3}{2}t}\left(\sin(\pi t)-t\pi\cos(\pi t)+c\right)}$$

Applying the initial condition gives,

$$0=y(1)=\tfrac{1}{2\pi^2}\mathbf{e}^{-\frac{3}{2}}(\pi+c) \quad \to c=-\pi \quad \Rightarrow \quad \boxed{y(t)=\tfrac{1}{2\pi^2}t^4\mathbf{e}^{-\frac{3}{2}t}\left(\sin(\pi t)-t\pi\cos(\pi t)-\pi\right)}$$

5. (2 pts) We know from Calc I that relative extrema occur at critical points and critical points are those points where the derivative is zero or doesn't exist. However, we've been told that for this problem the derivative exists and is continuous everywhere so that means that at the critical point that gives the relative minimum, *i.e.* at t_c, we have $y'(t_c)=0$ and $y(t_c)=3$ so all we need to do is plug t_c into the differential equation and then solve for t_c.

$$y'(t_c)+6y(t_c)=4+7\mathbf{e}^{t_c} \qquad \Rightarrow \qquad 6(3)=4+7\mathbf{e}^{t_c} \qquad \Rightarrow \qquad \boxed{t_c=\ln(2)}$$

7. (3 pts) Here's the solution to the differential equation. I'll leave the solution details to you.

$$\mu=\mathbf{e}^{-3t} \qquad y(t)=-1+\tfrac{7}{4}\mathbf{e}^{t}+c\mathbf{e}^{3t} \qquad \underline{y(t)=-1+\tfrac{7}{4}\mathbf{e}^{t}+\left(\tfrac{13}{4}-\alpha^2\right)\mathbf{e}^{3t}}$$

Now, the first two terms will always to go ∞ at $t\to\infty$ so any variation in the behavior will come from the third term. This behavior of the third term will depend on the value of α ,

$$\tfrac{13}{4}-\alpha^2<0 \qquad \left(\tfrac{13}{4}-\alpha^2\right)\mathbf{e}^{3t}\to-\infty \qquad\qquad \tfrac{13}{4}-\alpha^2>0 \qquad \left(\tfrac{13}{4}-\alpha^2\right)\mathbf{e}^{3t}\to\infty$$

$$\tfrac{13}{4}-\alpha^2=0 \qquad \left(\tfrac{13}{4}-\alpha^2\right)\mathbf{e}^{3t}=0$$

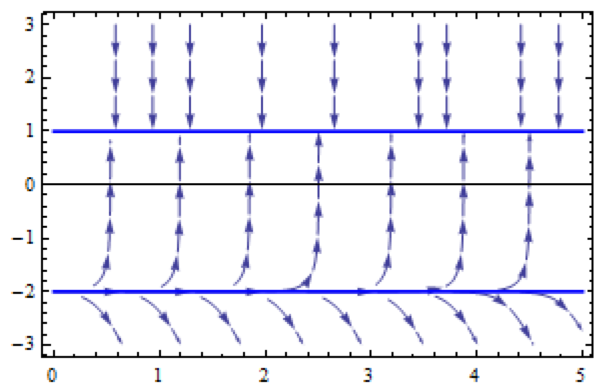
Also, the exponent on the third term is larger than that on the second term and so it will dominate any differences in long term behavior. So, the long term behavior (and dependence on α) is,

$$-\tfrac{\sqrt{13}}{2}\le\alpha\le\tfrac{\sqrt{13}}{2} \quad\Rightarrow\quad y(t)\to\infty \qquad\qquad \alpha<-\tfrac{\sqrt{13}}{2},\ \ \alpha>\tfrac{\sqrt{13}}{2} \quad\Rightarrow\quad y(t)\to-\infty$$

Not Graded

2. The derivative will be zero at $y=1$ and $y=-2$. A sketch of the direction field and some solutions is to the right. From this we can see the long term behavior and dependence on $y(0)$.

$$y(0)>-2 \qquad\qquad y(t)\to1$$
$$y(0)=-2 \qquad\qquad y(t)=-2$$
$$y(0)<-2 \qquad\qquad y(t)\to-\infty$$



3. Don't forget that the y' needs a coefficient of 1 and the *y* needs to be moved over to the left side before we do any of the work.

$$y'-\frac{2x}{x^2+4}y=\frac{x}{\sqrt{x^2+4}} \qquad\qquad \mu=\mathbf{e}^{-\int\frac{2x}{x^2+4}dx}=\mathbf{e}^{-\ln\left(x^2+4\right)}=\left(x^2+4\right)^{-1}$$

$$\int\left[\left(x^2+4\right)^{-1}y\right]dx=\int x\left(x^2+4\right)^{-\frac{3}{2}}dx \quad\Rightarrow\quad \left(x^2+4\right)^{-1}y=-\left(x^2+4\right)^{-\frac{1}{2}}+c$$

$$\Rightarrow\quad \underline{y(x)=-\sqrt{x^2+4}+c\left(x^2+4\right)}$$

Applying the initial condition gives,

$$8=-2+4c \quad\to\quad c=\tfrac{5}{2} \quad\Rightarrow\quad \boxed{y(x)=-\sqrt{x^2+4}+\tfrac{5}{2}\left(x^2+4\right)}$$

6. Here's the solution to the differential equation. I'll leave the solution details to you.

$$\mu=\mathbf{e}^{4t} \quad y(t)=-\tfrac{1}{16}+\tfrac{1}{4}t-\tfrac{1}{2}\mathbf{e}^{-2t}+c\mathbf{e}^{-4t} \qquad\qquad \underline{y(t)=-\tfrac{1}{16}+\tfrac{1}{4}t-\tfrac{1}{2}\mathbf{e}^{-2t}+\left(y_0+\tfrac{9}{16}\right)\mathbf{e}^{-4t}}$$

Now, we are in a similar situation to **#5.** We know *where* the relative minimum is and we can see that the solution and its derivative exist and is continuous everywhere. So, the relative minimum must occur at the critical point $t_c=0.5$ and we know that $y'(0.5)=0$ and we need to determine the value of $y(0.5)=y_{\min}$ let's plug $t_c=0.5$ into the differential equation and solve for $y_{\min}$.

$$y'(0.5)+4y(0.5)=0.5-\mathbf{e}^{-1} \quad\to\quad 4y_{\min}=\tfrac{1}{2}-\mathbf{e}^{-1} \quad\Rightarrow\quad y_{\min}=\tfrac{1}{8}-\tfrac{1}{4}\mathbf{e}^{-1}=0.03303$$

Now, all we need to do is plug $y(0.5) = \frac{1}{8} - \frac{1}{4}\mathbf{e}^{-1}$ into the solution and solve for y_0.

$$\frac{1}{8} - \frac{1}{4}\mathbf{e}^{-1} = -\frac{1}{16} + \frac{1}{8} - \frac{1}{2}\mathbf{e}^{-1} + \left(y_0 + \frac{9}{16}\right)\mathbf{e}^{-2} \qquad \Rightarrow \qquad \boxed{y_0 = \frac{1}{16}\mathbf{e}^{2} + \frac{1}{4}\mathbf{e} - \frac{9}{16} = 0.57889}$$