Math 3301 **Homework Set 2 – Solutions** **10 Points**

1. (2 pts) Not much to do other than separate, integrate and solve.

$$\int y^{-4}\,dy = \int \tfrac{\mathbf{e}^{2x}}{1+\mathbf{e}^{2x}}\,dx \quad \to \quad -\tfrac{1}{3}y^{-3} = \tfrac{1}{2}\ln\left(1+\mathbf{e}^{2x}\right)+c$$

Applying the initial condition gives,

$$\tfrac{1}{3} = \tfrac{1}{2}\ln\left(2\right)+c \quad \to \quad c = \tfrac{1}{3}-\tfrac{1}{2}\ln\left(2\right) \quad \to \quad \boxed{y\left(x\right) = \frac{1}{\sqrt[3]{\tfrac{3}{2}\ln\left(2\right)-1-\tfrac{3}{2}\ln\left(1+\mathbf{e}^{2x}\right)}}}$$

Now, for the interval of validity all we need to is worry about is division by zero since the cube root can have negative numbers under it. So, we need to avoid,

$$\tfrac{3}{2}\ln\left(2\right)-1-\tfrac{3}{2}\ln\left(1+\mathbf{e}^{2x}\right)=0 \quad \to \quad \ln\left(1+\mathbf{e}^{2x}\right)=0.02648 \quad \to \quad \mathbf{e}^{2x}=0.02684 \quad \to \quad x=-1.8090$$

So, we have two possible intervals of validity : $-\infty < x < -1.8090$, $\boxed{-1.8090 < x < \infty}$ and the second is the correct one because it is the one that contains $x = 0$.

3. (3 pts) Not much to do other than separate, integrate and solve.

$$\int 5+y\,dy = \int 1-8x\,dx \quad \to \quad \tfrac{1}{2}y^2+5y = x-4x^2+c$$

Applying the initial condition gives,

$$\tfrac{1}{2}-5=c \quad \to \quad c=-\tfrac{9}{2} \quad \to \quad \underline{\tfrac{1}{2}y^2+5y-\left(-4x^2+x-\tfrac{9}{2}\right)=0}$$

$$y\left(x\right)=\frac{-5\pm\sqrt{25+2\left(-4x^2+x-\frac{9}{2}\right)}}{2\left(\frac{1}{2}\right)}=-5\pm\sqrt{-8x^2+2x+16}$$

The initial condition tells us that the "+" is the correct sign and so the solution is,

$$\boxed{y\left(x\right)=-5+\sqrt{-8x^2+2x+16}}$$

For the interval of validity we just need to avoid negative values under the root.

$$x=\tfrac{1}{8}\left(1\pm\sqrt{129}\right)=-1.2947,\ 1.5447$$

The interval of validity is then : $-1.2947 \le x \le 1.5447$

5. (5 pts) Here's the two IVP's for the system.

$$Q_1' = \left(20\right)\left(8\right)-\frac{2Q_1}{800+6t}=160-\frac{Q_1}{400+3t} \qquad Q_1\left(0\right)=100$$

$$Q_2' = \left(12\right)\left(5\right)-\frac{5Q_2}{1000}=60-\frac{Q_2}{200} \qquad Q_2\left(t_c\right)=Q_1\left(t_c\right)$$

In the second IVP t_c is the time when the tank overflows. This is easy to compute.

$$1000=800+6t_c \qquad t_c=\tfrac{100}{3}$$

Also note that the volume in the 2nd IVP will be fixed at the volume of the tank. Here is the solution to the first IVP.

$$Q_1(t) = 40(400+3t) - \frac{15900\left(400^{\frac{1}{3}}\right)}{(400+3t)^{\frac{1}{3}}}$$

We can now completely write down the second IVP is then,

$$Q_2' = 60 - \frac{Q_2}{200} \qquad\qquad Q_2\left(\tfrac{100}{3}\right) = Q_1\left(\tfrac{100}{3}\right) = 5239.7475$$

Here is the solution to this IVP.

$$Q_2(t) = 12000 - 7986.2947\mathbf{e}^{-\frac{t}{200}}$$

To answer this we then need $Q_2\left(\frac{100}{3}+2\right) = Q_2\left(\frac{106}{3}\right) = \boxed{5307.0131}$. Remember this is 2 hours AFTER the second process started and so will be at $t = \frac{106}{3}$.

Not Graded

2. Not much to do other than separate, integrate and solve.

$$\int y^2\,dy = \int \tfrac{1}{6x-7}\,dx \quad \to \quad \tfrac{1}{3}y^3 = \tfrac{1}{6}\ln|6x-7| + c$$

Applying the initial condition gives,

$$\tfrac{8}{3} = \tfrac{1}{6}\ln(7) + c \quad \to \quad c = \tfrac{8}{3} - \tfrac{1}{6}\ln(7) \quad \to \quad \boxed{y(x) = \sqrt[3]{\tfrac{1}{2}\ln|6x-7| + 8 - \tfrac{1}{2}\ln(7)}}$$

Now, for the interval of validity all we have to worry about is that we can't take logarithms of zero and so we clearly have to avoid $x = \frac{6}{7}$. This means that we have two possible intervals of validity

$$\boxed{-\infty < x < \tfrac{6}{7}}, \quad \tfrac{6}{7} < x < \infty$$

As noted above the first interval is actual interval of validity because it contains $x = 0$.

4. Here's the IVP and the solution (I'll let you check the solution details).

$$Q' = 5\left(12 - 7\,\mathbf{e}^{-\frac{t}{25}}\right) - \frac{5Q}{250} \qquad\qquad Q(0) = 25$$

$$\boxed{Q(t) = 3000 + 1750\mathbf{e}^{-\frac{t}{25}} - 4725\mathbf{e}^{-\frac{t}{50}}} \qquad\qquad \lim_{t\to\infty} Q(t) = 3000$$

We can see from the limit that the equilibrium amount of salt would be 400 ounces.

6. Don't forget to shift the time to take into account that this situation starts at $t = \frac{106}{3}$.

$$Q_3' = (10)(5) - \frac{14Q_3}{1000 - 9\left(t - \frac{106}{3}\right)} = 50 - \frac{14Q_3}{1318 - 9t} \qquad\qquad Q_3\left(\tfrac{106}{3}\right) = 5307.0131$$