2. (2 pts)

$$W = \begin{vmatrix} x^{-3} & g \\ -3x^{-4} & g' \end{vmatrix} = x^{-3}g' + 3x^{-4}g = x^{-5}\mathbf{e}^{x} \qquad \to \qquad g' + \tfrac{3}{x}g = x^{-2}\mathbf{e}^{x}$$

This is a simple linear differential equation so I'll let you verify my solution.

$$\boxed{g(x) = x^{-2}\mathbf{e}^{x} - x^{-3}\mathbf{e}^{x} + cx^{-3}}$$

3. (2 pts) I'll leave it to you to verify that $y_c(t) = c_1\mathbf{e}^{-10t} + c_2 t\mathbf{e}^{-10t}$. The guess for the particular and its derivatives are,

$$Y_P = At^3 + Bt^2 + Ct + D \qquad Y_P' = 3At^2 + 2Bt + C \qquad Y_P'' = 6At + 2B$$

Plugging this into the differential equation and simplifying gives,

$$100At^3 + (60A + 100B)t^2 + (6A + 40B + 100C)t + 2B + 20C + 100D = 50t^3 - 4t$$

Setting coefficients equal and solving gives,

$$\begin{aligned} t^3 :& \quad 100A = 50 \\ t^2 :& \quad 60A + 100B = 0 \\ t^1 :& \quad 6A + 40B + 100C = -4 \\ t^0 :& \quad 2B + 20C + 100D = 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= \tfrac{1}{2} \\ B &= -\tfrac{3}{10} \\ C &= \tfrac{1}{20} \\ D &= -\tfrac{1}{250} \end{aligned}$$

The general solution is then,

$$\boxed{y(t) = c_1\mathbf{e}^{-10t} + c_2 t\mathbf{e}^{-10t} + \tfrac{1}{2}t^3 - \tfrac{3}{10}t^2 + \tfrac{1}{20}t - \tfrac{1}{250}}$$

6. (3 pts) I'll leave it to you to verify that $y_c(t) = c_1\mathbf{e}^{t} + c_2\mathbf{e}^{8t}$. The guess for the particular solution and its derivatives are,

$$Y_P = \mathbf{e}^{3t}\left[A\cos(t) + B\sin(t)\right] \quad Y_P' = \mathbf{e}^{3t}\left[(3A+B)\cos(t) + (-A+3B)\sin(t)\right]$$

$$Y_P'' = \mathbf{e}^{3t}\left[(8A+6B)\cos(t) + (-6A+8B)\sin(t)\right]$$

Plugging this into the differential equation and simplifying gives,

$$\mathbf{e}^{3t}\left[(-11A-3B)\cos(t) + (3A-11B)\sin(t)\right] = 5\mathbf{e}^{3t}\cos(t)$$

Setting coefficients equal and solving gives,

$$\begin{aligned} \mathbf{e}^{3t}\cos(t):& \quad -11A - 3B = 5 \\ \mathbf{e}^{3t}\sin(t):& \quad 3A - 11B = 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= -\tfrac{11}{26} \\ B &= -\tfrac{3}{26} \end{aligned}$$

The general solution is then,

$$\boxed{y(t) = c_1\mathbf{e}^{t} + c_2\mathbf{e}^{8t} + \mathbf{e}^{3t}\left[-\tfrac{11}{26}\cos(t) - \tfrac{3}{26}\sin(t)\right]}$$

7. (3 pts) I'll leave it to you to verify that $y_c(t) = c_1\cos(2t) + c_2\sin(2t)$. The guess for the particular solution and its derivatives are,

$$Y_P = \left(At^2 + Bt + C\right)\mathbf{e}^{6t} \qquad Y_P' = \left(6At^2 + (2A+6B)t + B + 6C\right)\mathbf{e}^{6t}$$

$$Y_P'' = \left(36At^2 + (24A+36B)t + 2A + 12B + 36C\right)\mathbf{e}^{6t}$$

Plugging this into the differential equation and simplifying gives,

$$\left(2A+12B+40C\right)\mathbf{e}^{6t}+\left(24A+40B\right)t\mathbf{e}^{6t}+40At^2\mathbf{e}^{6t}=200t^2\mathbf{e}^{6t}$$

Setting coefficients equal and solving gives,

$$\begin{aligned}\mathbf{e}^{6t}:&\quad 2A+12B+40C=0 & & A=5\\ t\mathbf{e}^{6t}:&\quad 24A+40B=0 &\Rightarrow\quad & B=-3\\ t^2\mathbf{e}^{6t}:&\quad 40A=200 & & C=\tfrac{13}{20}\end{aligned}$$

The general solution is,

$$\underline{y(t)=c_1\cos(2t)+c_2\sin(2t)+\left(5t^2-3t+\tfrac{13}{20}\right)\mathbf{e}^{6t}}$$

Now apply the initial conditions.

$$\begin{aligned}-1=y(0)=c_1+\tfrac{13}{20}&\quad\Rightarrow\quad & c_1=-\tfrac{33}{20}\\ -3=y'(0)=2c_2+\tfrac{9}{10}& & c_2=-\tfrac{39}{20}\end{aligned}$$

The actual solution is then,

$$\boxed{y(t)=-\tfrac{33}{20}\cos(2t)-\tfrac{39}{20}\sin(2t)+\left(5t^2-3t+\tfrac{13}{20}\right)\mathbf{e}^{6t}}$$

Not Graded

1. Compute the Wronskian.

$$W(y_1,y_2)=\begin{vmatrix}\mathbf{e}^{r_1t} & \mathbf{e}^{r_2t}\\ r_1\mathbf{e}^{r_1t} & r_2\mathbf{e}^{r_2t}\end{vmatrix}=r_2\mathbf{e}^{(r_1+r_2)t}-r_1\mathbf{e}^{(r_1+r_2)t}=(r_2-r_1)\mathbf{e}^{(r_1+r_2)t}\neq 0\quad\text{b/c}\quad r_2\neq r_1$$

So they are a fundamental set of solutions and the general solution is what I claimed it to be and the general solution is $y(t)=c_1\mathbf{e}^{r_1t}+c_2\mathbf{e}^{r_2t}$.

4. I'll leave it to you to verify that $y_c(t)=c_1\mathbf{e}^{-\frac{1}{2}t}\cos\left(\sqrt{5}\,t\right)+c_2\mathbf{e}^{-\frac{1}{2}t}\sin\left(\sqrt{5}\,t\right)$. The guess for the particular solution and its derivatives are,

$$Y_P=A\mathbf{e}^{9t}\qquad Y_P'=9A\mathbf{e}^{9t}\qquad Y_P''=81\mathbf{e}^{9t}$$

Plugging this into the differential equation and simplifying gives,

$$381A=-4\mathbf{e}^{9t}$$

Setting coefficients equal and solving gives : $381A=-4\quad\Rightarrow\quad A=-\frac{4}{381}$

The general solution is then : $\boxed{y(t)=c_1\mathbf{e}^{-\frac{1}{2}t}\cos\left(\sqrt{5}\,t\right)+c_2\mathbf{e}^{-\frac{1}{2}t}\sin\left(\sqrt{5}\,t\right)-\frac{4}{381}\mathbf{e}^{9t}}$

5. I'll leave it to you to verify that $y_c(t)=c_1\mathbf{e}^{-4t}+c_2\mathbf{e}^{7t}$. The guess for the particular solution and its derivatives are,

$$Y_P=A\cos(3t)+B\sin(3t)\quad Y_P'=-3A\sin(3t)+3B\cos(3t)\quad Y_P''=-9A\cos(3t)-9B\sin(3t)$$

Plugging this into the differential equation and simplifying gives,

$$\left(-37A-9B\right)\cos\left(3t\right)+\left(9A-37B\right)\sin\left(3t\right)=20\sin\left(3t\right)$$

Setting coefficients equal and solving gives,

$$\begin{aligned}\cos\left(3t\right)&:\quad -37A-9B=0\\ \sin\left(3t\right)&:\quad 9A-37B=20\end{aligned}\qquad\Rightarrow\qquad\begin{aligned}A&=\tfrac{18}{145}\\ B&=-\tfrac{74}{145}\end{aligned}$$

The general solution is then,

$$\boxed{y\left(t\right)=c_1\mathbf{e}^{-4t}+c_2\mathbf{e}^{7t}+\tfrac{18}{145}\cos\left(3t\right)-\tfrac{74}{145}\sin\left(3t\right)}$$