Math 3301 **Homework Set 6 – Solutions** **10 Points**

1. (3 pts) I'll leave it to you to verify that the complimentary solution is : $\underline{y_c(t) = c_1 + c_2\mathbf{e}^{-9t}}$. The guess for the particular solution (note the addition of the *t* on the first term to account for the fact that this term would be in the complimentary solution otherwise) and its derivatives are,

$$Y_P = At + B\cos(3t) + D\sin(3t)$$

$$Y_P' = A - 3B\sin(3t) + 3D\cos(3t) \qquad Y_P'' = -9B\cos(3t) - 9D\sin(3t)$$

Plugging this into the differential equation and simplifying gives,

$$9A + (-9B + 27D)\cos(3t) + (-27B - 9D)\sin(3t) = 6 + 2\cos(3t) - 9\sin(3t)$$

Setting coefficients equal gives,

$$\begin{aligned} t^0 &: & 9A &= 6 & & & A &= \tfrac{2}{3} \\ \cos(3t) &: & -9B + 27D &= 2 & &\Rightarrow & B &= \tfrac{5}{18} \\ \sin(3t) &: & -27B - 9D &= -9 & & & C &= \tfrac{1}{6} \end{aligned}$$

The general solution is then,

$$\boxed{y(t) = c_1 + c_2\mathbf{e}^{-9t} + \tfrac{2}{3}t + \tfrac{5}{18}\cos(3t) + \tfrac{1}{6}\sin(3t)}$$

3. (4 pts) I'll leave it to you to verify that the complimentary solution is : $y_c(t) = c_1\mathbf{e}^{-2t} + c_2\mathbf{e}^{6t}$. The guess for the particular solution (note the addition of the *t* on the third term to account for the fact that this term would be in the complimentary solution otherwise) and its derivatives are,

$$Y_P = At + B + Ct\mathbf{e}^{-2t} \qquad Y_P' = A + C\mathbf{e}^{-2t} - 2Ct\mathbf{e}^{-2t} \qquad Y_P'' = -4C\mathbf{e}^{-2t} + 4Ct\mathbf{e}^{-2t}$$

Plugging this into the differential equation and simplifying gives,

$$-12At - 4A - 12B - 8C\mathbf{e}^{-2t} = 6t - 8\mathbf{e}^{-2t}$$

Setting coefficients equal and solving gives,

$$\begin{aligned} t^1 &: & -12A &= 6 & & & A &= -\tfrac{1}{2} \\ t^0 &: & -4A - 12B &= 0 & &\Rightarrow & B &= \tfrac{1}{6} \\ \mathbf{e}^{3t} &: & -8C &= -8 & & & C &= 1 \end{aligned}$$

The general solution is then : $\underline{y(t) = c_1\mathbf{e}^{-2t} + c_2\mathbf{e}^{6t} - \tfrac{1}{2}t + \tfrac{1}{6} + t\mathbf{e}^{-2t}}$

Now apply the initial conditions.

$$\begin{aligned} 6 = y(0) &= c_1 + c_2 + \tfrac{1}{6} & &\Rightarrow & c_1 &= \tfrac{61}{48} \\ -1 = y'(0) &= -2c_1 + 6c_2 + \tfrac{1}{2} & & & c_2 &= \tfrac{73}{16} \end{aligned}$$

The actual solution is then : $\boxed{y(t) = \tfrac{61}{48}\mathbf{e}^{-2t} + \tfrac{73}{16}\mathbf{e}^{6t} - \tfrac{1}{2}t + \tfrac{1}{6} + t\mathbf{e}^{-2t}}$

6. (3 pts) I'll leave it to you to verify that $y_c(t) = c_1\mathbf{e}^{-t}\cos(2t) + c_2\mathbf{e}^{-t}\sin(2t)$. Here's the information we need for Variation of Parameters and don't forget to divide by a 9 to get the coefficient of the 2nd derivative to by one.

$$W = \begin{vmatrix} \mathbf{e}^{-t}\cos(2t) & \mathbf{e}^{-t}\sin(2t) \\ -\mathbf{e}^{-t}\cos(2t) - 2\mathbf{e}^{-t}\sin(2t) & -\mathbf{e}^{-t}\sin(2t) + 2\mathbf{e}^{-t}\cos(2t) \end{vmatrix}$$

$$= 2\mathbf{e}^{-2t}\cos^2(2t) + 2\mathbf{e}^{-2t}\sin^2(2t) = \underline{2\mathbf{e}^{-2t}} \qquad\qquad g(t) = 15\mathbf{e}^{-t}$$

The particular solution is then,

$$Y_P = -\mathbf{e}^{-t}\cos(2t)\int \frac{\mathbf{e}^{-t}\sin(2t)\left(15\mathbf{e}^{-t}\right)}{2\mathbf{e}^{-2t}}\,dt + \mathbf{e}^{-t}\sin(2t)\int \frac{\mathbf{e}^{-t}\cos(2t)\left(15\mathbf{e}^{-t}\right)}{2\mathbf{e}^{-2t}}\,dt$$

$$= -\tfrac{1}{2}\mathbf{e}^{-t}\cos(2t)\int 15\sin(2t)\,dt + \tfrac{1}{2}\mathbf{e}^{-t}\sin(2t)\int 15\cos(2t)\,dt$$

$$= \tfrac{15}{4}\mathbf{e}^{-t}\cos^2(2t) + \tfrac{15}{4}\mathbf{e}^{-t}\sin^2(2t) = \underline{\tfrac{15}{4}\mathbf{e}^{-t}}$$

The general solution is then : $\boxed{y(t) = c_1\mathbf{e}^{-t}\cos(2t) + c_2\mathbf{e}^{-t}\sin(2t) + \tfrac{15}{4}\mathbf{e}^{-t}}$

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2. I'll leave it to you to verify that the complimentary solution is : $\underline{y_c(t) = c_1\mathbf{e}^{3t} + c_2 t\mathbf{e}^{3t}}$. The guess for the particular solution (note the addition of the t^2 in the second term to account for the fact that this term is in the complimentary solution without it) and its derivatives are,

$$Y_P = A\mathbf{e}^{-t} + Bt^2\mathbf{e}^{3t} \qquad\qquad Y_P' = -A\mathbf{e}^{-t} + 2Bt\mathbf{e}^{3t} + 3Bt^2\mathbf{e}^{3t}$$

$$Y_P'' = A\mathbf{e}^{-t} + 2B\mathbf{e}^{3t} + 12Bt\mathbf{e}^{3t} + 9Bt^2\mathbf{e}^{3t}$$

Plugging this into the differential equation and simplifying gives,

$$16A\mathbf{e}^{-t} + 2B\mathbf{e}^{3t} = \mathbf{e}^{-t} + 4\mathbf{e}^{3t}$$

Setting coefficients equal gives,

$$\begin{aligned} \mathbf{e}^{-t}:&\quad 16A = 1 \\ \mathbf{e}^{3t}:&\quad 2B = 4 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= \tfrac{1}{16} \\ B &= 2 \end{aligned}$$

The general solution is then : $\boxed{y(t) = c_1\mathbf{e}^{3t} + c_2 t\mathbf{e}^{3t} + \tfrac{1}{16}\mathbf{e}^{-t} + 2t^2\mathbf{e}^{3t}}$

4. I'll leave it to you to verify that $y_c(t) = c_1\mathbf{e}^{-8t} + c_2\mathbf{e}^{7t}$. The guess for the particular solution is,

$$Y_P = t(At + B)\mathbf{e}^{-8t} + C\mathbf{e}^{12t} + Dt + E$$

Note that we needed to tack a *t* onto the first term to prevent it from being in the complimentary solution.

5. I'll leave it to you to verify that $y_c(t) = c_1\mathbf{e}^{-5t}\cos(2t) + c_2\mathbf{e}^{-5t}\sin(2t)$. The guess for the particular solution is,

$$Y_P = (At+B)\cos(2t) + (Ct+D)\sin(2t) + t\mathbf{e}^{-5t}(Et+F)\cos(2t) + t\mathbf{e}^{-5t}(Gt+H)\sin(2t)$$

Note that we needed to tack a *t* onto the last two portions to prevent them from being in the complimentary solution.

7. I'll leave it to you to verify that $y_c(t) = c_1\mathbf{e}^{-\frac{1}{3}t} + c_2\mathbf{e}^{\frac{1}{3}t}$. Here's the information we need for Variation of Parameters.

$$W = \begin{vmatrix} \mathbf{e}^{-\frac{1}{3}t} & \mathbf{e}^{\frac{1}{3}t} \\ -\frac{1}{3}\mathbf{e}^{-\frac{1}{3}t} & \frac{1}{3}\mathbf{e}^{\frac{1}{3}t} \end{vmatrix} = \frac{2}{3} \qquad g(t) = \tfrac{10}{9} - \tfrac{1}{3}t$$

Don't forget to divide by the 9 to get a coefficient of 1 on the second derivative term. The particular solution is then,

$$\begin{aligned} Y_P &= -\mathbf{e}^{-\frac{1}{3}t}\int \frac{\mathbf{e}^{\frac{1}{3}t}\left(\frac{10}{9} - \frac{1}{3}t\right)}{\frac{2}{3}}\,dt + \mathbf{e}^{\frac{1}{3}t}\int \frac{\mathbf{e}^{-\frac{1}{3}t}\left(\frac{10}{9} - \frac{1}{3}t\right)}{\frac{2}{3}}\,dt \\ &= -\mathbf{e}^{-\frac{1}{3}t}\int \mathbf{e}^{\frac{1}{3}t}\left(\tfrac{5}{3} - \tfrac{1}{2}t\right)dt + \mathbf{e}^{\frac{1}{3}t}\int \mathbf{e}^{-\frac{1}{3}t}\left(\tfrac{5}{3} - \tfrac{1}{2}t\right)dt \\ &= -\tfrac{1}{2}\mathbf{e}^{-\frac{1}{3}t}\mathbf{e}^{\frac{1}{3}t}(19-3t) + \tfrac{1}{2}\mathbf{e}^{\frac{1}{3}t}\mathbf{e}^{-\frac{1}{3}t}(3t-1) = \underline{3t-10} \end{aligned}$$

The general solution is,

$$\underline{y(t) = c_1\mathbf{e}^{-\frac{1}{3}t} + c_2\mathbf{e}^{\frac{1}{3}t} + 3t - 10}$$

Now apply the initial conditions.

$$\begin{aligned} 0 &= y(0) = c_1 + c_2 - 10 \\ 8 &= y'(0) = -\tfrac{1}{3}c_1 + \tfrac{1}{3}c_2 + 3 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} c_1 &= -\tfrac{5}{2} \\ c_2 &= \tfrac{25}{2} \end{aligned}$$

The actual solution is then : $\boxed{y(t) = -\tfrac{5}{2}\mathbf{e}^{-\frac{1}{3}t} + \tfrac{25}{2}\mathbf{e}^{\frac{1}{3}t} + 3t - 10}$

8. I'll leave it to you to verify most of the solution work. The key quantities for the problem are,

$$m = \tfrac{4}{32} = \tfrac{1}{8} \qquad L = \tfrac{8}{12} = \tfrac{2}{3} \qquad k = \frac{4}{2/3} = 6 \qquad \omega_0 = \sqrt{\frac{6}{1/8}} = \sqrt{48} = 4\sqrt{3} = 6.9282$$

The IVP is,

$$\tfrac{1}{8}u'' + 6u = 0 \qquad u(0) = -\tfrac{4}{12} = -\tfrac{1}{3} \quad u'(0) = -\tfrac{10}{12} = -\tfrac{5}{6}$$

The general solution is,

$$u(t) = c_1\cos\left(4\sqrt{3}\,t\right) + c_2\sin\left(4\sqrt{3}\,t\right)$$

Applying the initial conditions gives,

$$\underline{u(t) = -\tfrac{1}{3}\cos\left(4\sqrt{3}\,t\right) - \tfrac{5}{24\sqrt{3}}\sin\left(4\sqrt{3}\,t\right)}$$

Now, reduce down to a single cosine.

$$R = \sqrt{\left(-\tfrac{1}{3}\right)^2 + \left(-\tfrac{5}{24\sqrt{3}}\right)^2} = 0.3544 \quad \delta_1 = \tan^{-1}\left(\frac{-\frac{5}{24\sqrt{3}}}{-\frac{1}{3}}\right) = 0.3463 \quad \delta_2 = \delta_1 + \pi = 3.4879$$

In this case the second angle is the correct one and so the single cosine form of the solution is,

$$\boxed{u(t) = 0.3544\cos\left(4\sqrt{3}\,t - 3.4879\right)}$$

9. I'll be leaving it to you to verify most of the solution work. The key quantities for the problem are,

$$m = 2 \qquad L = 0.4 \qquad k = \tfrac{(2)(9.8)}{0.4} = 49 \qquad \gamma = \tfrac{15}{0.75} = 20 \qquad \gamma_{CR} = 2\sqrt{(49)(2)} = 14\sqrt{2} = 19.7990$$

We have over damping in this case. The IVP is then,

$$2u'' + 20u' + 49u = 0 \qquad u(0) = 0.5 \quad u'(0) = 0.15$$

The general solution is,

$$u(t) = c_1\mathbf{e}^{\frac{1}{2}\left(-10-\sqrt{2}\right)t} + c_2\mathbf{e}^{\frac{1}{2}\left(-10+\sqrt{2}\right)t} = c_1\mathbf{e}^{-5.7071t} + c_2\mathbf{e}^{-4.2929t}$$

Applying the initial conditions gives : $\boxed{u(t) = -1.6238\mathbf{e}^{-5.7071t} + 2.1238\mathbf{e}^{-4.2929t}}$

10. Taking what we can from **#8** we have the following,

$$\tfrac{1}{8}u'' + 6u = 0 \quad u(0) = -\tfrac{1}{3},\ u'(0) = -\tfrac{5}{6} \qquad \underline{u_c(t) = c_1\cos\left(4\sqrt{3}\,t\right) + c_2\sin\left(4\sqrt{3}\,t\right)}$$

Note that because $\omega = 5 \neq 4\sqrt{3} = \omega_0$ we will **NOT** have resonance. Undetermined Coefficients will probably be the easiest for a particular solution so,

$$U_P = A\cos(5t) + B\sin(5t) \quad \rightarrow \quad \tfrac{23}{8}A\cos(5t) + \tfrac{23}{8}B\sin(5t) = 6\cos(5t) - \sin(5t)$$

Setting coefficients equal and solving gives $A = \frac{48}{23}$, $B = -\frac{8}{23}$. The general solution is then,

$$u(t) = c_1\cos\left(4\sqrt{3}\,t\right) + c_2\sin\left(4\sqrt{3}\,t\right) + \tfrac{48}{23}\cos(5t) - \tfrac{8}{23}\sin(5t)$$

Applying the initial conditions gives,

$$u(t) = -\tfrac{167}{69}\cos\left(4\sqrt{3}\,t\right) + \tfrac{125}{552\sqrt{3}}\sin\left(4\sqrt{3}\,t\right) + \tfrac{48}{23}\cos(5t) - \tfrac{8}{23}\sin(5t)$$

Now, reduce the first sine/cosine pair down to a single cosine.

$$R = \sqrt{\left(-\tfrac{167}{69}\right)^2 + \left(\tfrac{125}{552\sqrt{3}}\right)^2} = 2.4238 \qquad \delta_1 = \tan^{-1}\left(\frac{125/552\sqrt{3}}{-167/69}\right) = -0.0540 \qquad \delta_2 = \delta_1 + \pi = 3.0876$$

In this case the second angle is correct. Next reduce the second sine/cosine pair down to a single cosine.

$$R = \sqrt{\left(\tfrac{48}{23}\right)^2 + \left(-\tfrac{8}{23}\right)^2} = 2.1157 \qquad \delta_1 = \tan^{-1}\left(\frac{-8/23}{48/23}\right) = -0.1651 \qquad \delta_2 = \delta_1 + \pi = 2.9765$$

In this case the first angle is correct. The action solution is then,

$$\boxed{u(t) = 2.4238\cos\left(4\sqrt{3}\,t - 3.0876\right) + 2.1157\cos(5t + 0.1651)}$$

11. Taking what we can from **#9** we have the following,

$$2u'' + 20u' + 49u = 0 \quad u(0) = 0.5 \ u'(0) = 0.15 \qquad u_c(t) = c_1\mathbf{e}^{-5.7071t} + c_2\mathbf{e}^{-4.2929t}$$

Undetermined Coefficients will probably be the easiest for a particular solution so,

$$U_P = A\cos(2t) + B\sin(2t)$$

$$(41A + 40B)A\cos(2t) + (41B - 40A)B\sin(2t) = 8\sin(2t)$$

Setting coefficients equal and solving gives $A = -\frac{320}{3281}$, $B = \frac{328}{3281}$. The general solution is then,

$$u(t) = c_1\mathbf{e}^{-5.7071t} + c_2\mathbf{e}^{-4.2929t} - \tfrac{320}{3281}\cos(2t) + \tfrac{328}{3281}\sin(2t)$$

Applying the initial conditions gives,

$$u(t) = -1.7785\mathbf{e}^{-5.7071t} + 2.3761\mathbf{e}^{-4.2929t} - \tfrac{320}{3281}\cos(2t) + \tfrac{328}{3281}\sin(2t)$$

Now reduce the sine/cosine down to a single cosine.

$$R = \sqrt{\left(-\tfrac{320}{3281}\right)^2 + \left(\tfrac{328}{3281}\right)^2} = 0.1397 \quad \delta_1 = \tan^{-1}\left(\frac{328/3281}{-320/3281}\right) = -0.7977 \quad \delta_2 = \delta_1 + \pi = 2.3439$$

In this case the second angle is correct. The actual solution is then,

$$\boxed{u(t) = -1.7785\mathbf{e}^{-5.7071t} + 2.3761\mathbf{e}^{-4.2929t} + 0.1397\cos(2t - 2.3439)}$$