2. (3 pts) Take the Laplace transform of everything.

$$s^2Y(s)-sy(0)-y'(0)+2\left(sY(s)-y(0)\right)+3Y(s)=\frac{\mathbf{e}^{-4s}}{s-4}-\frac{7\mathbf{e}^{-10s}}{s}$$

$$\left(s^2+2s+3\right)Y(s)=\frac{\mathbf{e}^{-4s}}{s-4}-\frac{7\mathbf{e}^{-10s}}{s}$$

$$Y(s)=\frac{\mathbf{e}^{-4s}}{(s-4)\left(s^2+2s+3\right)}-\frac{7\mathbf{e}^{-10s}}{s\left(s^2+2s+3\right)}$$

$$Y(s)=\mathbf{e}^{-4s}F(s)-7\mathbf{e}^{-10s}G(s)$$

I'll leave it to you to verify the partial fraction work.

$$F(s)=\frac{1}{27}\left[\frac{1}{s-4}-\frac{s+6}{(s+1)^2+2}\right]=\frac{1}{27}\left[\frac{1}{s-4}-\frac{s+1}{(s+1)^2+2}-\frac{5\frac{\sqrt{2}}{\sqrt{2}}}{(s+1)^2+2}\right]$$

$$\underline{f(t)=\tfrac{1}{27}\left[\mathbf{e}^{4t}-\mathbf{e}^{-t}\cos\left(\sqrt{2}\,t\right)-\tfrac{5}{\sqrt{2}}\mathbf{e}^{-t}\sin\left(\sqrt{2}\,t\right)\right]}$$

$$G(s)=\frac{1}{3}\left[\frac{1}{s}-\frac{s+1+1}{(s+1)^2+2}\right]=\frac{1}{3}\left[\frac{1}{s}-\frac{s+1}{(s+1)^2+2}-\frac{1\frac{\sqrt{2}}{\sqrt{2}}}{(s+1)^2+2}\right]$$

$$\underline{g(t)=\tfrac{1}{3}\left[1-\mathbf{e}^{-t}\cos\left(\sqrt{2}\,t\right)-\tfrac{1}{\sqrt{2}}\mathbf{e}^{-t}\sin\left(\sqrt{2}\,t\right)\right]}$$

The solution is then,

$$Y(s)=\mathbf{e}^{-4s}F(s)-7\mathbf{e}^{-10s}G(s) \qquad \boxed{y(t)=u_4(t)f(t-4)-7u_{10}(t)g(t-10)}$$

where $f(t)$ and $g(t)$ are shown above.

5. (4 pts) Take the Laplace transform of everything.

$$2\left[s^2Y(s)-sy(0)-y'(0)\right]-7\left(sY(s)-y(0)\right)=2\mathbf{e}^{-s}+\frac{14\mathbf{e}^{-2s}}{s-3}$$

$$\left(2s^2-7s\right)Y(s)=2\mathbf{e}^{-s}+\frac{14\mathbf{e}^{-2s}}{s-3}$$

$$Y(s)=\frac{2\mathbf{e}^{-s}}{s(2s-7)}+\frac{14\mathbf{e}^{-2s}}{s(s-3)(2s-7)}$$

$$Y(s)=2\mathbf{e}^{-s}F(s)+14\mathbf{e}^{-2s}G(s)$$

I'll leave it to you to verify the partial fraction work.

$$F(s)=\frac{-\frac{1}{7}}{s}+\frac{\frac{2}{7}}{2s-7} \qquad \to \qquad \underline{f(t)=-\tfrac{1}{7}+\tfrac{1}{7}\mathbf{e}^{\frac{7}{2}t}}$$

$$G(s)=\frac{\frac{1}{21}}{s}-\frac{\frac{1}{3}}{s-3}+\frac{\frac{4}{7}}{2s-7} \qquad \to \qquad \underline{g(t)=\tfrac{1}{21}-\tfrac{1}{3}\mathbf{e}^{3t}+\tfrac{2}{7}\mathbf{e}^{\frac{7}{2}t}}$$

The solution is then,

$$Y(s) = 2\mathbf{e}^{-s}F(s) + 14\mathbf{e}^{-2s}G(s) \qquad \boxed{y(t) = 2u_1(t)f(t-1) + 14u_2(t)g(t-2)}$$

where $f(t)$ and $g(t)$ are shown above.

8. (3 pts) Take the Laplace transform of everything.

$$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + Y(s) = G(s)$$

$$(s^2 - 2s + 1)Y(s) - 9s + 14 = G(s)$$

$$Y(s) = \frac{G(s)}{(s-1)^2} + \frac{9s-14}{(s-1)^2}$$

$$Y(s) = G(s)F(s) + H(s)$$

I'll leave it to you to verify the partial fractions.

$$F(s) = \frac{1}{(s-1)^2} \quad \rightarrow \quad \underline{f(t) = t\mathbf{e}^{-t}}$$

$$H(s) = \frac{9}{s-1} - \frac{5}{(s-1)^2} \quad \rightarrow \quad \underline{h(t) = 9\mathbf{e}^{-t} - 5t\mathbf{e}^{-t}}$$

The solution is then,

$$Y(s) = G(s)F(s) + H(s) \quad \Rightarrow \quad \boxed{y(t) = \int_0^t \tau\mathbf{e}^{-\tau}g(t-\tau)\,d\tau + 9\mathbf{e}^{-t} - 5t\mathbf{e}^{-t}}$$

Not Graded

1. Take the Laplace transform of everything.

$$2\left[s^2Y(s) - sy(0) - y'(0)\right] + 7(sY(s) - y(0)) - 4Y(s) = \frac{6\mathbf{e}^{-7s}}{s+1}$$

$$(2s^2 + 7s - 4)Y(s) + 2s + 7 = \frac{6\mathbf{e}^{-7s}}{s+1}$$

$$Y(s) = \frac{6\mathbf{e}^{-7s}}{(s+1)(2s-1)(s+4)} - \frac{2s+7}{(2s-1)(s+4)}$$

$$Y(s) = 6\mathbf{e}^{-7s}F(s) - G(s)$$

I'll leave it to you to verify the partial fraction work.

$$F(s) = \frac{-\frac{1}{9}}{s+1} + \frac{\frac{1}{27}}{s+4} + \frac{\frac{4}{27}}{2s+1} \quad \rightarrow \quad \underline{f(t) = -\tfrac{1}{9}\mathbf{e}^{-t} + \tfrac{1}{27}\mathbf{e}^{-4t} + \tfrac{2}{27}\mathbf{e}^{-\frac{1}{2}t}}$$

$$G(s) = \frac{\frac{1}{9}}{s+4} + \frac{\frac{16}{9}}{2s+1} \quad \rightarrow \quad \underline{g(t) = \tfrac{1}{9}\mathbf{e}^{-4t} + \tfrac{8}{9}\mathbf{e}^{-\frac{1}{2}t}}$$

The solution is then,

$$Y(s) = 6\mathbf{e}^{-7s}F(s) - G(s) \qquad \boxed{y(t) = 6u_7(t)f(t-7) - g(t)}$$

where $f(t)$ and $g(t)$ are shown above.

3. Take the Laplace transform of everything.

$$s^2Y(s)-sy(0)-y'(0)+8\left(sY(s)-y(0)\right)+16Y(s)=\frac{4\mathbf{e}^{-8s}}{s^2}-\frac{7}{s^2}$$

$$\left(s^2+8s+16\right)Y(s)-5=\frac{4\mathbf{e}^{-8s}-7}{s^2}$$

$$Y(s)=\frac{4\mathbf{e}^{-8s}-7}{s^2(s+4)^2}+\frac{5}{(s+4)^2}$$

$$Y(s)=\left(4\mathbf{e}^{-8s}-7\right)F(s)+G(s)$$

I'll leave it to you to verify the partial fraction work.

$$F(s)=\frac{1}{32}\left[\frac{-1}{s}+\frac{2}{s^2}+\frac{1}{s+4}+\frac{2}{(s+4)^2}\right]\quad\rightarrow\quad \underline{f(t)=\tfrac{1}{32}\left[-1+2t+\mathbf{e}^{-4t}+2t\mathbf{e}^{-4t}\right]}$$

$$G(s)=\frac{5}{(s+4)^2}\quad\rightarrow\quad \underline{g(t)=5t\mathbf{e}^{-4t}}$$

The solution is then,

$$Y(s)=4\mathbf{e}^{-8s}F(s)-7F(s)+G(s)\qquad \boxed{y(t)=4u_8(t)f(t-8)-7f(t)+g(t)}$$

where $f(t)$ and $g(t)$ are shown above.

4. Take the Laplace transform of everything.

$$s^2Y(s)-sy(0)-y'(0)-3\left(sY(s)-y(0)\right)-18Y(s)=9\mathbf{e}^{-8s}$$

$$\left(s^2-3s-18\right)Y(s)+4s-11=9\mathbf{e}^{-8s}$$

$$Y(s)=\frac{9\mathbf{e}^{-8s}}{(s-6)(s+3)}+\frac{11-4s}{(s-6)(s+3)}$$

$$Y(s)=9\mathbf{e}^{-8s}F(s)+G(s)$$

I'll leave it to you to verify the partial fraction work.

$$F(s)=\frac{\frac{1}{9}}{s-6}-\frac{\frac{1}{9}}{s+3}\quad\rightarrow\quad \underline{f(t)=\tfrac{1}{9}\mathbf{e}^{6t}-\tfrac{1}{9}\mathbf{e}^{-3t}}$$

$$G(s)=\frac{-\frac{13}{9}}{s-6}-\frac{\frac{23}{9}}{s+3}\quad\rightarrow\quad \underline{g(t)=-\tfrac{13}{9}\mathbf{e}^{6t}-\tfrac{23}{9}\mathbf{e}^{-3t}}$$

The solution is then,

$$Y(s)=9\mathbf{e}^{-8s}F(s)+G(s)\qquad \boxed{y(t)=9u_8(t)f(t-8)+g(t)}$$

where $f(t)$ and $g(t)$ are shown above.

6. This looks like a convolution integral using $g\left(t\right)=t^2$ and $h\left(t\right)=\mathbf{e}^{-4t}$ (make sure to pay attention to signs on this term!) so the Laplace transform is then,

$$F\left(s\right)=G\left(s\right)H\left(s\right)=\left(\frac{2}{s^3}\right)\left(\frac{1}{s+4}\right)=\boxed{\frac{2}{s^3\left(s+4\right)}}$$

7. Rewrite the transform as,

$$H\left(s\right)=\frac{6}{s+1}\frac{1}{s-4}=F\left(s\right)G\left(s\right)\qquad\Rightarrow\qquad f\left(t\right)=6\mathbf{e}^{-t},\quad g\left(t\right)=\mathbf{e}^{4t}$$

Then,

$$h\left(t\right)=\int_0^t f\left(\tau\right)g\left(t-\tau\right)d\tau=\int_0^t 6\mathbf{e}^{-\tau}\mathbf{e}^{4(t-\tau)}d\tau=\int_0^t 6\mathbf{e}^{4t-5\tau}d\tau=2=-\tfrac{6}{5}\mathbf{e}^{4t-5\tau}\Big|_0^t=\boxed{\tfrac{6}{5}\left(\mathbf{e}^{4t}-\mathbf{e}^{-t}\right)}$$