8. (3 pts) I'll leave it to you to verify that the eigenvalues and eigenvectors for this matrix are,

$$\lambda_1 = 5 \quad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda_2 = 3 \quad \vec{\eta}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

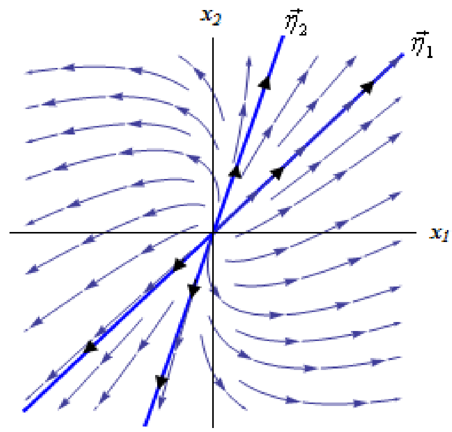
Applying the initial conditions gives,

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} c_1 + c_2 = 2 \\ c_1 + 3c_2 = -1 \end{matrix} \quad \Rightarrow \quad \begin{matrix} c_1 = \frac{7}{2} \\ c_2 = -\frac{3}{2} \end{matrix}$$

The actual solution is then,

$$\boxed{\vec{x}(t) = \tfrac{7}{2} \mathbf{e}^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \tfrac{3}{2} \mathbf{e}^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

A sketch of the phase portrait is to the right. The equilibrium solution in this case is an **unstable node**.



10. (4 pts) First convert to a system in matrix form.

$$\begin{matrix} x_1 = y \\ x_2 = y' \end{matrix} \quad \Rightarrow \quad \begin{matrix} x_1' = x_2 \\ x_2' = -5x_1 + 2x_2 \end{matrix} \quad \Rightarrow \quad \vec{x}' = \begin{bmatrix} 0 & 1 \\ -5 & 2 \end{bmatrix} \vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$$

I'll leave it to you to verify that the eigenvalues/eigenvectors are,

$$\lambda_1 = 1 + 2i \quad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 1 + 2i \end{pmatrix} \qquad \lambda_2 = 1 - 2i \quad \vec{\eta}^{(2)} = \begin{pmatrix} 1 \\ 1 - 2i \end{pmatrix}$$

Next,

$$\begin{aligned} \mathbf{e}^{(1+2i)t} \begin{pmatrix} 1 \\ 1 + 2i \end{pmatrix} &= \mathbf{e}^{t} \left(\cos(2t) + i \sin(2t) \right) \begin{pmatrix} 1 \\ 1 + 2i \end{pmatrix} \\ &= \mathbf{e}^{t} \begin{pmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix} + i \mathbf{e}^{t} \begin{pmatrix} \sin(2t) \\ \sin(2t) + 2\cos(2t) \end{pmatrix} \end{aligned}$$

The general solution is then,

$$\underline{\vec{x}(t) = c_1 \mathbf{e}^{t} \begin{pmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix} + c_2 \mathbf{e}^{t} \begin{pmatrix} \sin(2t) \\ \sin(2t) + 2\cos(2t) \end{pmatrix}}$$

Applying the initial conditions gives,

$$\begin{pmatrix} 4 \\ -6 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} c_1 = 4 \\ c_1 + 2c_2 = -6 \end{matrix} \quad \Rightarrow \quad \begin{matrix} c_1 = 4 \\ c_2 = -5 \end{matrix}$$

The actual solution to the system is then,

$$\vec{x}(t) = 4\mathbf{e}^{t}\begin{pmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix} - 5\mathbf{e}^{t}\begin{pmatrix} \sin(2t) \\ \sin(2t) + 2\cos(2t) \end{pmatrix} = \mathbf{e}^{t}\begin{pmatrix} 4\cos(2t) - 5\sin(2t) \\ -6\cos(2t) - 13\sin(2t) \end{pmatrix}$$

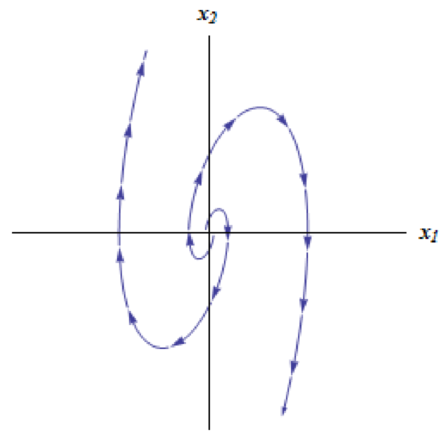
The solution to the original 2nd order differential equation is then,

$$y(t) = \mathbf{e}^{t}\left(4\cos(2t) - 5\sin(2t)\right)$$

From the system,

$$\begin{bmatrix} 0 & 1 \\ -5 & 2 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

we can see that trajectories will cross the x_1 axis going downwards. A sketch of the phase portrait is to the right. The equilibrium solution in this case is an **asymptotically stable spiral**.



12. (3 pts) I'll leave it to you to verify that the eigenvalues and eigenvectors for this matrix are,

$$\lambda_{1,2} = -10 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Next,

$$\begin{bmatrix} -6 & -12 \\ 3 & 6 \end{bmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \Rightarrow \quad 3\rho_1 + 6\rho_2 = 1 \quad \Rightarrow \quad \rho_1 = \tfrac{1}{3} - 2\rho_2 \quad \Rightarrow \quad \vec{\rho} = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \quad \text{using } \rho_2 = 0$$

The general solution is then,

$$\vec{x}(t) = c_1\mathbf{e}^{-10t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2\left\{t\,\mathbf{e}^{-10t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \mathbf{e}^{-10t}\begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}\right\}$$

Applying the initial conditions gives,

$$\begin{pmatrix} 9 \\ 20 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \begin{aligned} -2c_1 + \tfrac{1}{3}c_2 &= 9 \\ c_1 &= 20 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} c_1 &= 20 \\ c_2 &= 147 \end{aligned}$$

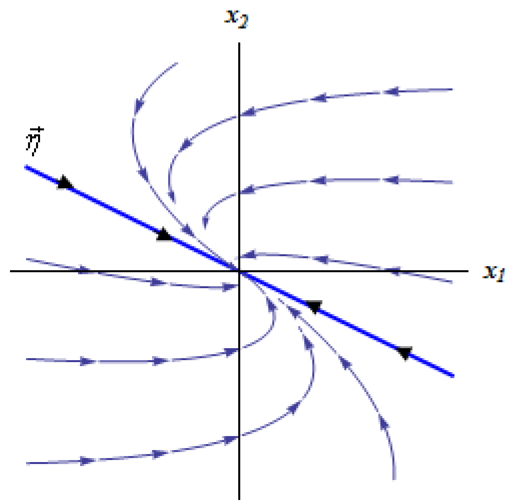
The actual solution is then,

$$\vec{x}(t) = 20\mathbf{e}^{-10t}\begin{pmatrix}-2\\1\end{pmatrix} + 147\left\{t\,\mathbf{e}^{-10t}\begin{pmatrix}-2\\1\end{pmatrix} + \mathbf{e}^{-10t}\begin{pmatrix}\frac{1}{3}\\0\end{pmatrix}\right\}$$
$$= \mathbf{e}^{-10t}\begin{pmatrix}9\\20\end{pmatrix} + 147t\,\mathbf{e}^{-10t}\begin{pmatrix}-2\\1\end{pmatrix}$$

From the system,

$$\begin{bmatrix}-16 & -12\\3 & -4\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-16\\3\end{pmatrix}$$

we can see that trajectories will cross the x_1 axis going upwards. A sketch of the phase portrait is to the right. The equilibrium solution in this case is an **asymptotically stable improper node**.



Not Graded

1. Here's the determinant we'll need for the eigenvalues.

$$\det(A-\lambda I) = \begin{vmatrix}5-\lambda & 1\\-1 & 7-\lambda\end{vmatrix} = \lambda^2 - 12\lambda + 36 = (\lambda-6)^2 = 0 \quad\Rightarrow\quad \lambda_{1,2} = 6$$

So, it looks like we've got a double eigenvalue. Now, find the eigenvector.

$$\lambda_{1,2} = 6: \qquad \begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix}\begin{pmatrix}\eta_1\\\eta_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad\Rightarrow\quad -\eta_1 + \eta_2 = 0 \quad\Rightarrow\quad \eta_2 = \eta_1$$

$$\vec{\eta}^{(1)} = \begin{pmatrix}1\\1\end{pmatrix} \quad \text{using } \eta_1 = 1$$

2. Here's the determinant we'll need for the eigenvalues.

$$\det(A-\lambda I) = \begin{vmatrix}-4-\lambda & -6\\3 & 2-\lambda\end{vmatrix} = \lambda^2 + 2\lambda + 10 = 0 \quad\Rightarrow\quad \lambda_{1,2} = -1 \pm 3i$$

Here is the eigenvector for the first eigenvalue.

$$\lambda_1 = -1+3i: \quad \begin{bmatrix}3-3i & -6\\3 & 3-3i\end{bmatrix}\begin{pmatrix}\eta_1\\\eta_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad\Rightarrow\quad 3\eta_1 + (3-3i)\eta_2 = 0 \quad\Rightarrow\quad \eta_1 = (-1+i)\eta_2$$

$$\vec{\eta}^{(1)} = \begin{pmatrix}-1+i\\1\end{pmatrix} \quad \text{using } \eta_2 = 1$$

Then by the fact from class the eigenvector for $\lambda_2 = -1-3i$ will be,

$$\vec{\eta}^{(2)} = \begin{pmatrix}-1-i\\1\end{pmatrix}$$

3. Here's the determinant we'll need for the eigenvalues.

$$\det\left(A-\lambda I\right)=\begin{vmatrix}\frac{4}{3}-\lambda & -\frac{2}{9}\\ 3 & -1-\lambda\end{vmatrix}=\lambda^2-\tfrac{1}{3}\lambda-\tfrac{2}{3}=\left(\lambda-1\right)\left(\lambda+\tfrac{2}{3}\right)=0 \quad\Rightarrow\quad \lambda_1=1,\ \lambda_2=-\tfrac{2}{3}$$

Here are the eigenvectors for each eigenvalue.

$$\lambda_1=1:\ \begin{bmatrix}\frac{1}{3} & -\frac{2}{9}\\ 3 & -2\end{bmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad\Rightarrow\quad 3\eta_1-2\eta_2=0 \quad\Rightarrow\quad \eta_1=\tfrac{2}{3}\eta_2$$

$$\vec{\eta}^{(1)}=\begin{pmatrix}2\\ 3\end{pmatrix} \quad \text{using } \eta_2=3$$

$$\lambda_2=-\tfrac{2}{3}:\ \begin{bmatrix}2 & -\frac{2}{9}\\ 3 & -\frac{1}{3}\end{bmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad\Rightarrow\quad 3\eta_1-\tfrac{1}{3}\eta_2=0 \quad\Rightarrow\quad \eta_1=\tfrac{1}{9}\eta_2$$

$$\vec{\eta}^{(2)}=\begin{pmatrix}1\\ 9\end{pmatrix} \quad \text{using } \eta_2=9$$

4.

$$\begin{array}{lll} x_1=y & x_1'=x_2 & x_1\left(0\right)=3\\ x_2=y' & x_2'=\frac{5}{9}x_1-\frac{1}{9}x_2 & x_2\left(0\right)=10\end{array}$$

$$\vec{x}'=\begin{bmatrix}0 & 1\\ \frac{5}{9} & -\frac{1}{9}\end{bmatrix}\vec{x} \qquad \vec{x}\left(0\right)=\begin{pmatrix}3\\ 10\end{pmatrix}$$

5.

$$\begin{array}{lll} x_1=y & x_1'=x_2 & x_1\left(0\right)=10\\ x_2=y' & x_2'=x_3 & x_2\left(0\right)=-20\\ x_3=y'' & x_3'=x_4 & x_3\left(0\right)=30\\ x_4=y''' & x_4'=-9x_1-7x_3+2x_4 & x_4\left(0\right)=-40\end{array}$$

$$\vec{x}'=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -9 & 0 & -7 & 2\end{bmatrix}\vec{x} \qquad \vec{x}\left(0\right)=\begin{pmatrix}10\\ -20\\ 30\\ -40\end{pmatrix}$$

6. I'll leave it to you to verify that the (messy) eigenvalues and eigenvectors for this matrix are,

$$\lambda_1=\tfrac{1}{2}\left(9+\sqrt{29}\right) \quad \vec{\eta}^{(1)}=\begin{pmatrix}5+\sqrt{29}\\ 14\end{pmatrix} \qquad \lambda_2=\tfrac{1}{2}\left(9-\sqrt{29}\right) \quad \vec{\eta}^{(2)}=\begin{pmatrix}5-\sqrt{29}\\ 14\end{pmatrix}$$

The general solution is then,

$$\boxed{\vec{x}\left(t\right)=c_1\mathbf{e}^{\frac{1}{2}\left(9+\sqrt{29}\right)t}\begin{pmatrix}1+\sqrt{29}\\ 14\end{pmatrix}+c_2\mathbf{e}^{\frac{1}{2}\left(9-\sqrt{29}\right)t}\begin{pmatrix}1-\sqrt{29}\\ 14\end{pmatrix}}$$

Don't get excited when you run into "messy" eigenvalues and/or eigenvectors. They are fairly common.

7. I'll leave it to you to verify that the eigenvalues and eigenvectors for this matrix are,

$$\lambda_1 = -9 \quad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \lambda_2 = 2 \quad \vec{\eta}^{(2)} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

The general solution is then,

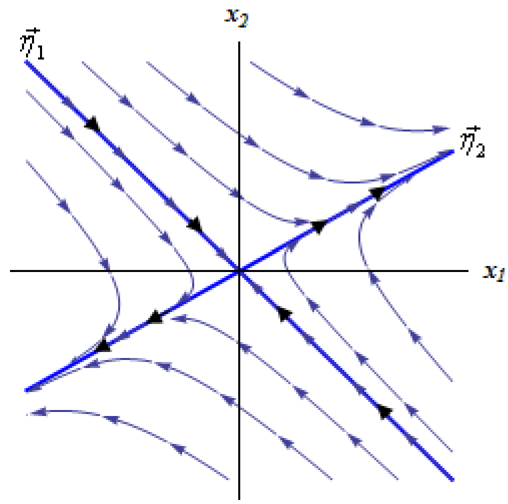
$$\vec{x}(t) = c_1 \mathbf{e}^{-9t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{2t} \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

Applying the initial conditions gives,

$$\begin{pmatrix} 0 \\ 6 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 7 \\ 4 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{r} -c_1 + 7c_2 = 0 \\ c_1 + 4c_2 = 6 \end{array} \quad \Rightarrow \quad \begin{array}{l} c_1 = \frac{42}{11} \\ c_2 = \frac{6}{11} \end{array}$$

The actual solution is then,

$$\boxed{\vec{x}(t) = \tfrac{42}{11} \mathbf{e}^{-9t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \tfrac{6}{11} \mathbf{e}^{2t} \begin{pmatrix} 7 \\ 4 \end{pmatrix}}$$



A sketch of the phase portrait is to the right. The equilibrium solution in this case is an **unstable saddle point**.

9. I'll leave it to you to verify that the eigenvalues and eigenvectors for this matrix are,

$$\lambda_1 = 4i \quad \vec{\eta}^{(1)} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} \qquad \lambda_2 = -4i \quad \vec{\eta}^{(2)} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$$

Next,

$$\mathbf{e}^{(4i)t} \begin{pmatrix} 1-i \\ 1 \end{pmatrix} = \left(\cos(4t) + i\sin(4t)\right) \begin{pmatrix} 1-i \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(4t) + \sin(4t) \\ \cos(4t) \end{pmatrix} + i \begin{pmatrix} \sin(4t) - \cos(4t) \\ \sin(4t) \end{pmatrix}$$

The general solution is then,

$$\underline{\vec{x}(t) = c_1 \begin{pmatrix} \cos(4t) + \sin(4t) \\ \cos(4t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(4t) - \cos(4t) \\ \sin(4t) \end{pmatrix}}$$

Applying the initial conditions gives,

$$\begin{pmatrix} 12 \\ 0 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{r} c_1 - c_2 = 12 \\ c_1 = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} c_1 = 0 \\ c_2 = -12 \end{array}$$

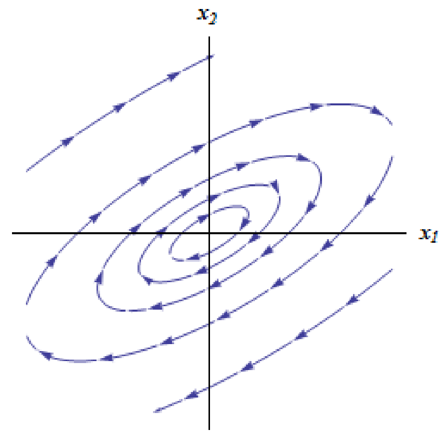
The actual solution is then,

$$\boxed{\vec{x}(t) = \begin{pmatrix} 12\cos(4t) - 12\sin(4t) \\ -12\sin(4t) \end{pmatrix}}$$

From the system,

$$\begin{bmatrix} -4 & 8 \\ -4 & 4 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$$

we can see that trajectories will cross the x_1 axis going downwards. A sketch of the phase portrait is to the right. The equilibrium solution in this case is a **stable center**.



11. I'll leave it to you to verify that the eigenvalues and eigenvectors for this matrix are,

$$\lambda_{1,2} = 1 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Next,

$$\begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad 2\rho_1 - 2\rho_2 = 1 \quad \Rightarrow \quad \rho_1 = \tfrac{1}{2} + \rho_2 \quad \Rightarrow \quad \vec{\rho} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \quad \text{using } \rho_2 = 0$$

The general solution is then,

$$\underline{\vec{x}(t) = c_1\mathbf{e}^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\left\{t\,\mathbf{e}^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbf{e}^{t}\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}\right\}}$$

Applying the initial conditions gives,

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} c_1 + \tfrac{1}{2}c_2 &= 0 \\ c_1 &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 1 \\ c_2 &= -2 \end{aligned}$$

The actual solution is then,

$$\boxed{\vec{x}(t) = \mathbf{e}^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2\left\{t\,\mathbf{e}^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbf{e}^{t}\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}\right\} = \mathbf{e}^{t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - 2t\,\mathbf{e}^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

From the system,

$$\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

we can see that trajectories will cross the x_1 axis going upwards. A sketch of the phase portrait is to the right. The equilibrium solution in this case is an **unstable improper node**.