1. (3 pts) $t=\frac{1}{2}\sec\theta \quad dt=\frac{1}{2}\sec\theta\tan\theta d\theta \quad \left(4t^2-1\right)^{\frac{3}{2}}=\left(\sec^2\theta-1\right)^{\frac{3}{2}}=\left(\tan^2\theta\right)^{\frac{3}{2}}=\left|\tan\theta\right|^3$

$t=\frac{1}{2}:\ \frac{1}{2}=\frac{1}{2}\sec\theta \ \rightarrow\ \cos\theta=1 \ \rightarrow\ \theta=0 \qquad z=2:\ 2=\frac{1}{2}\sec\theta \ \rightarrow\ \cos\theta=\frac{1}{4} \ \rightarrow\ \theta=\cos^{-1}\left(\frac{1}{4}\right)=1.3181$

In this range of θ (1st quadrant)we can see that tangent is positive and so $\left(4t^2-1\right)^{\frac{3}{2}}=\tan^3\theta$. The integral is,

$$\begin{aligned}\int_{\frac{1}{2}}^{2} t^5\left(4t^2-1\right)^{\frac{3}{2}}dt &= \int_0^{1.3181}\tfrac{1}{32}\sec^5\theta\tan^3\theta\left(\tfrac{1}{2}\sec\theta\tan\theta\right)d\theta\\ &=\tfrac{1}{64}\int_0^{1.3181}\sec^4\theta\tan^4\theta\ \sec^2\theta\,d\theta\\ &=\tfrac{1}{64}\int_0^{1.3181}\left(\tan^2\theta+1\right)^2\tan^4\theta\ \sec^2\theta\,d\theta \qquad u=\tan\theta\\ &=\tfrac{1}{64}\int_0^{1.3181}u^8+2u^6+u^4\,du=\tfrac{1}{64}\left(\tfrac{1}{9}\tan^9\theta+\tfrac{2}{7}\tan^7\theta+\tfrac{1}{5}\tan^5\theta\right)\Big|_0^{1.3181}=\boxed{401.2449}\end{aligned}$$

4. (2 pts)

$$\sin z=4\tan\theta \quad \cos z\,dz=4\sec^2\theta\,d\theta \quad \sqrt{16+\sin^2 z}=\sqrt{16+16\tan^2\theta}=4\left|\sec\theta\right|=4\sec\theta$$

$$\int\cos z\sqrt{16+\sin^2 z}\,dz=\int 16\sec^3\theta\,d\theta=8\left(\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right)+c$$

A quick right triangle shows us that : $\tan\theta=\frac{\sin z}{4}$, $\sec\theta=\frac{\sqrt{16+\sin^2 z}}{4}$ and the integral becomes,

$$\int\cos z\sqrt{16+\sin^2 z}\,dz=\boxed{8\left(\frac{\sin z\sqrt{16+\sin^2 z}}{16}+\ln\left|\frac{\sin z+\sqrt{16+\sin^2 z}}{4}\right|\right)+c}$$

7. (2 pts)

$$\frac{3t^2+1}{(t-1)(t+2)^2}=\frac{A}{t-1}+\frac{B}{t+2}+\frac{C}{(t+2)^2} \quad\Rightarrow\quad 3t^2+1=A(t+2)^2+B(t-1)(t+2)+C(t-1)$$

Picking values of *t* gives,

$$\begin{aligned} t=1:&\quad 4=9A & & A=\tfrac{4}{9}\\ t=-2:&\quad 13=-3C & \Rightarrow\quad & B=\tfrac{23}{9}\\ t=0:&\quad 1=4A-2B-C & & C=-\tfrac{13}{3}\end{aligned}$$

The integral is then,

$$\int\frac{3t^2+1}{(t-1)(t+2)^2}dt=\int\frac{\frac{4}{9}}{t-1}+\frac{\frac{23}{9}}{t+2}-\frac{\frac{13}{3}}{(t+2)^2}dt=\boxed{\tfrac{4}{9}\ln\left|t-1\right|+\tfrac{23}{9}\ln\left|t+2\right|+\frac{\frac{13}{3}}{t+2}+c}$$

8. (3 pts)

$$\frac{4-z}{z^2\left(z^2+16\right)}=\frac{A}{z}+\frac{B}{z^2}+\frac{Cz+D}{z^2+16} \quad\rightarrow\quad 4-z=Az\left(z^2+16\right)+B\left(z^2+16\right)+z^2\left(Cz+D\right)$$

$$=\left(A+C\right)z^3+\left(B+D\right)z^2+16Az+16B$$

Setting coefficients equal gives,

$$\begin{array}{llcl} z^3: & A+C=0 & & A=-\frac{1}{16} \\ z^2: & B+D=0 & \Rightarrow & B=\frac{1}{4} \\ z^1: & 16A=-1 & & C=\frac{1}{16} \\ z^0: & 16B=4 & & D=-\frac{1}{4} \end{array}$$

The integral is then,

$$\int \frac{4-z}{z^2\left(z^2+16\right)}\,dz = \int \frac{-\frac{1}{16}}{z} + \frac{\frac{1}{4}}{z^2} + \frac{\frac{1}{16}z-\frac{1}{4}}{z^2+16}\,dz = \int \frac{-\frac{1}{16}}{z} + \frac{\frac{1}{4}}{z^2} + \frac{\frac{1}{16}z}{z^2+16} - \frac{\frac{1}{4}}{z^2+16}\,dz$$

$$= \boxed{-\tfrac{1}{16}\ln|z| - \frac{\frac{1}{4}}{z} + \tfrac{1}{32}\ln\left|z^2+16\right| - \tfrac{1}{16}\tan^{-1}\left(\tfrac{z}{4}\right) + c}$$

Not Graded

2. $x=\frac{\sqrt{2}}{3}\sin\theta \quad dz=\frac{\sqrt{2}}{3}\cos\theta\,d\theta \quad \sqrt{2-9x^2}=\sqrt{2-2\sin\theta^2}=\sqrt{2}\sqrt{\cos^2\theta}=\sqrt{2}\left|\cos\theta\right|=\sqrt{2}\cos\theta$

We are working an indefinite integral and so we can assume cosine is positive. The integral is,

$$\int \frac{x^2}{\sqrt{2-9x^2}}\,dx = \int \frac{\frac{2}{9}\sin^2\theta}{\sqrt{2}\cos\theta}\frac{\sqrt{2}}{3}\cos\theta\,d\theta = \tfrac{2}{27}\int \sin^2\theta\,d\theta = \tfrac{1}{27}\int 1-\cos(2\theta)\,d\theta$$

$$= \tfrac{1}{27}\left(\theta - \tfrac{1}{2}\sin(2\theta)\right) + c = \tfrac{1}{27}\left(\theta - \sin(\theta)\cos(\theta)\right) + c$$

Note the use of the double angle formula to get the result in to one that we can use to convert back to x's. A quick right triangle shows us that : $\cos\theta = \frac{\sqrt{2-9x^2}}{\sqrt{2}}$ and we can use any of the 6 inverse trig functions for the θ in the first term. The integral becomes,

$$\int \frac{x^2}{\sqrt{2-9x^2}}\,dx = \frac{1}{27}\left(\sin^{-1}\left(\frac{3x}{\sqrt{2}}\right) - \left(\frac{3x}{\sqrt{2}}\right)\left(\frac{\sqrt{2-9x^2}}{\sqrt{2}}\right)\right) + c$$

$$= \boxed{\frac{1}{27}\left(\sin^{-1}\left(\frac{3x}{\sqrt{2}}\right) - \frac{3x\sqrt{2-9x^2}}{2}\right) + c}$$

3. First,

$$9x^2-18x+13 = 9\left(x^2-2x+\tfrac{13}{9}\right) = 9\left(x^2-2x+1-1+\tfrac{13}{9}\right) = 9\left(\left(x-1\right)^2+\tfrac{4}{9}\right) = 9\left(x-1\right)^2+4$$

The integral is then,

$$\int \left(x-1\right)^3\sqrt{9x^2-18x+13}\,dx = \int \left(x-1\right)^3\sqrt{9\left(x-1\right)^2+4}\,dx$$

Now : $x-1=\frac{2}{3}\tan\theta \qquad dx=\frac{2}{3}\sec^2\theta \qquad \sqrt{9\left(x-1\right)^2+4}=\sqrt{4\tan^2\theta+4}=2\left|\sec\theta\right|=2\sec\theta$

$$\begin{aligned}\int (x-1)^3\sqrt{9x^2-18x+13}\,dx &= \int \tfrac{8}{27}\tan^3\theta\left(2\sec\theta\right)\left(\tfrac{2}{3}\sec^2\theta\right)dx\\ &= \tfrac{32}{81}\int\left(\sec^2\theta-1\right)\sec^2\theta\tan\theta\sec\theta\,dx\\ &= \tfrac{32}{81}\int\left(u^2-1\right)u^2\,du = \tfrac{32}{81}\left(\tfrac{1}{5}\sec^5\theta-\tfrac{1}{3}\sec^3\theta\right)+c\end{aligned}$$

A quick right triangle shows us that : $\sec\theta = \frac{\sqrt{9(x-1)^2+4}}{2}$ and the integral becomes,

$$\int (x-1)^3\sqrt{9x^2-18x+13}\,dx = \boxed{\tfrac{1}{405}\left(9x^2-18x+13\right)^{\frac{5}{2}} - \tfrac{4}{243}\left(9x^2-18x+13\right)^{\frac{3}{2}}+c}$$

5.

$$f(x) = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x^4}+\frac{E}{4-3x}+\frac{Fx+G}{x^2-8x+13}+\frac{Hx+I}{\left(x^2-8x+13\right)^2}$$

6. $\dfrac{4-7x}{(x+2)(3x-2)} = \dfrac{A}{x+2}+\dfrac{B}{3x-2} \qquad \Rightarrow \qquad 4-7x = A(3x-2)+B(x+2)$

Picking values of *x* gives,

$$\begin{aligned} x=-2:&\quad 18=-8A \\ x=\tfrac{2}{3}:&\quad -\tfrac{2}{3}=\tfrac{8}{3}B\end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A&=-\tfrac{9}{4}\\ B&=-\tfrac{1}{4}\end{aligned}$$

The integral is then,

$$\begin{aligned}\int_{-1}^{0}\frac{4-7x}{(x+2)(3x-2)}\,dx &= \int_{-1}^{0}\frac{-\frac{9}{4}}{x+2}-\frac{\frac{1}{4}}{3x-2}\,dx = \left(-\tfrac{9}{4}\ln|x+2|-\tfrac{1}{12}\ln|3x-2|\right)\Big|_{-1}^{0}\\ &= \boxed{-\tfrac{7}{3}\ln 2+\tfrac{1}{12}\ln 5 = -1.4832}\end{aligned}$$

9. In this case we first need to do long division on the integrand because the degree of the numerator is greater than the degree of the denominator. I'll leave it to you to verify that,

$$\frac{x^4}{(x-4)\left(x^2+3\right)} = 4+x+\frac{13x^2+48}{(x-4)\left(x^2+3\right)}$$

The integral is then,

$$\int\frac{x^4}{(x-4)\left(x^2+3\right)}\,dx = \int 4+x+\frac{13x^2+48}{(x-4)\left(x^2+3\right)}\,dx = \int 4+x\,dx+\int\frac{13x^2+48}{(x-4)\left(x^2+3\right)}\,dx$$

Doing partial fractions on the second integrand gives,

$$\frac{13x^2+48}{(x-4)\left(x^2+3\right)} = \frac{A}{x-4}+\frac{Bx+C}{x^2+3} \qquad \to \qquad \begin{aligned}13x^2+48 &= A\left(x^2+3\right)+(Bx+C)(x-4)\\ &= (A+B)x^2+(C-4B)x+3A-4C\end{aligned}$$

Setting coefficients equal gives,

$$\begin{aligned} x^2: &\quad A+B=13 & & A=\tfrac{256}{19} \\ x^1: &\quad C-4B=0 & \Rightarrow \quad & B=-\tfrac{9}{19} \\ x^0: &\quad 3A-4C=48 & & C=-\tfrac{36}{19} \end{aligned}$$

The integral is then (and note that I factored a $\frac{1}{19}$ out of each coefficient into the front of the integral),

$$\begin{aligned} \int \frac{x^4}{(x-4)(x^2+3)}\,dx &= \int 4+x\,dx + \tfrac{1}{19}\int \frac{256}{x-4} - \frac{9x+36}{x^2+3}\,dx \\ &= \int 4+x\,dx + \tfrac{1}{19}\int \frac{256}{x-4} - \frac{9x}{x^2+3} - \frac{36}{x^2+3}\,dx \\ &= \boxed{4x+\tfrac{1}{2}x^2+\tfrac{1}{19}\left(256\ln\left|x-4\right| - \tfrac{9}{2}\ln\left|x^2+3\right| - \tfrac{36}{\sqrt{3}}\tan^{-1}\left(\tfrac{x}{\sqrt{3}}\right)\right)+c} \end{aligned}$$