1. (3 pts) We'll need to find values of *y* here.

$$ds = \sqrt{1+\left[\tfrac{dx}{dy}\right]^2}\,dy = \sqrt{1+16y^2}\,dy \qquad L = \int_0^2 \sqrt{1+16y^2}\,dy$$

$$x = 1: 1 = 1+2y^2 \;\rightarrow\; y = 0 \qquad x = 9: 9 = 1+2y^2 \;\rightarrow\; y = 2$$

This will require a trig substitution.

$$y = \tfrac{1}{4}\tan\theta \quad dy = \tfrac{1}{4}\sec^2\theta\,d\theta \quad \sqrt{1+16y^2} = \sqrt{1+\tan^2\theta} = \left|\sec\theta\right|$$

$$y = 0: 0 = \tfrac{1}{4}\tan\theta \;\rightarrow\; \theta = \tan^{-1}(0) = 0 \qquad y = 2: 2 = \tfrac{1}{4}\tan\theta \;\rightarrow\; \theta = \tan^{-1}(8) = 1.4464$$

$$L = \int_0^2 \sqrt{1+16y^2}\,dy = \tfrac{1}{4}\int_{0-}^{1.4464} \sec^3\theta\,d\theta = \tfrac{1}{8}\left(\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right)\Big|_0^{1.4464} = \boxed{8.4093}$$

4. (2 pts) This integral will require a simple Calc I substitution so I'll leave it to you to verify the details.

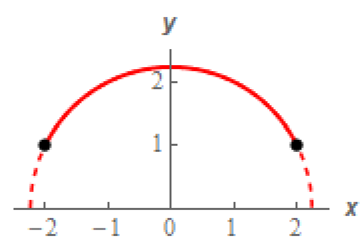
$$\tfrac{dy}{dx} = 8x \quad ds = \sqrt{1+9x^2}\,dx$$

$$A = \int_{-1}^{2} 2\pi x\sqrt{1+9x^2}\,dx = \tfrac{2}{27}\pi\left(1+9x^2\right)^{\frac{3}{2}}\Big|_{-1}^{2} = \boxed{\tfrac{2}{27}\pi\left[37^{\frac{3}{2}} - 10^{\frac{3}{2}}\right] = 45.0154}$$

8. (3 pts) First eliminate the parameter.

$$\cos\left(\tfrac{1}{3}t\right) = \tfrac{1}{2}x \quad \sin^2\left(\tfrac{1}{3}t\right) = \tfrac{1}{4}\left(y^2-1\right)$$

$$1 = \cos^2\left(\tfrac{1}{3}t\right) + \sin^2\left(\tfrac{1}{3}t\right) = \tfrac{1}{4}x^2 + \tfrac{1}{4}\left(y^2-1\right) \;\rightarrow\; \underline{x^2+y^2=5}$$



So, we'll have a portion of an circle, but not all of it. From the parametric equations we see the following limits on *x* and *y*.

$$-2 \le x \le 2 \qquad 1 \le y \le \sqrt{5}$$

We will then get the portion of the circle for these ranges of *x* and *y*. In addition, because *x* and *y* both contain sine or cosine the curve will oscillate back and forth as shown to the right. Remember it MUST stay on the solid line and so can *x*/*y* only oscillate if the curve also oscillates.

We will be at the left/right points when,

$$\text{left } (x=-2): \cos\left(\tfrac{1}{3}t\right) = -1 \;\rightarrow\; \tfrac{1}{3}t = \pi + 2\pi n \;\rightarrow\; t = 3\pi + 6\pi n$$

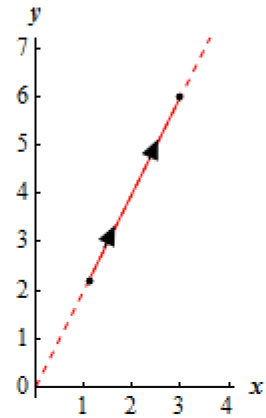
$$\text{right } (x=2): \cos\left(\tfrac{1}{3}t\right) = 1 \;\rightarrow\; \tfrac{1}{3}t = 0 + 2\pi n \;\rightarrow\; t = 6\pi n$$

So, it looks like the "easiest" range of *t*'s for one trace will be : $0 \le t \le 3\pi$.

9. (2 pts) This one is kind of tricky, but a quick rewrite on the *y* (using basic log properties) will make it really easy.

$$x = \ln(t), \quad y = 2\ln(t) = 2x$$

Because natural log is an increasing function both *x* and *y* can only increase as *t* increases and so this curve will trace out exactly once in the direction shown in the sketch to the right and it will trace out exactly once for the given range of *t*.



Ranges for *x* and *y* are,

$$\ln(3) \le x \le \ln(10) \qquad\qquad \ln(9) \le y \le \ln(100)$$

Not Graded

2. First get the *ds* for this problem.

$$\frac{dy}{dx} = \frac{-2x}{1-x^2} \qquad ds = \sqrt{1+\frac{4x}{\left(1-x^2\right)^2}}\,dx = \sqrt{\frac{\left(1-x^2\right)^2+4x}{\left(1-x^2\right)^2}}\,dx$$

$$= \sqrt{\frac{1+2x^2+x^4}{\left(1-x^2\right)^2}}\,dx = \sqrt{\frac{\left(1+x^2\right)^2}{\left(1-x^2\right)^2}}\,dx = \frac{1+x^2}{1-x^2}\,dx$$

We don't need absolute value bars after taking the root because the numerator is always positive and the denominator will be positive for the limits on *x*. The integral is then

$$L = \int_0^{1/2} \frac{1+x^2}{1-x^2}\,dx$$

To do this integral we'll need to do long division and then integration by parts. I'll leave the details to you to check.

$$\frac{1+x^2}{1-x^2} = -1+\frac{2}{1-x^2} = -1+\frac{2}{(1-x)(1+x)} = -1+\frac{1}{1+x}+\frac{1}{1-x} = -1+\frac{1}{x+1}-\frac{1}{x-1}$$

You can use either of the final forms of the partial fractions depending on which you are more comfortable with.

The length is then,

$$L = \int_0^{1/2} \frac{1+x^2}{1-x^2}\,dx = \int_0^{1/2} -1 + \frac{1}{x+1} - \frac{1}{x-1}\,dx$$

$$= \left(-x + \ln\left|x+1\right| - \ln\left|x-1\right|\right)\Big|_0^{1/2} = \boxed{-\tfrac{1}{2} + \ln\left(\tfrac{3}{2}\right) - \ln\left(\tfrac{1}{2}\right) = 0.5986}$$

3. This is not as difficult as it looks. This is just an ellipse and so we know that the limits on *x* and *y* are,

$$-\tfrac{1}{2} \le x \le \tfrac{1}{2} \qquad\qquad -2 \le y \le 2$$

So, we'll have limits once we get the integral set up. To do that we only need to solve for *x* or *y* and in this case it doesn't really matter which we solve for. I'll do solve for *y* since that is what most are used to solving for and it will actually be the easier one to work with in this case.

$$y = \pm\sqrt{4\left(1-4x^2\right)} = \pm 2\sqrt{1-4x^2}$$

The "+" will give the upper portion of the ellipse and the "-" will give the lower. Also since we can only use a single equation to find the arc length and because ellipses are symmetric we can find the length of the top and then just double that to get the complete length. This will also mean that we'll use *x* limits of integration. Here's the rest of the work.

$$\frac{dy}{dx} = \left(1-4x^2\right)^{-\frac{1}{2}}\left(-8x\right) = -\frac{8x}{\sqrt{1-4x^2}}$$

$$\sqrt{1+\left(\tfrac{dy}{dx}\right)^2} = \sqrt{1+\tfrac{64x^2}{1-4x^2}} = \sqrt{\tfrac{1+60x^2}{1-4x^2}} \qquad \Rightarrow \qquad \boxed{L = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\tfrac{1+60x^2}{1-4x^2}}\,dx}$$

5. Here's the work for the *ds*

$$\tfrac{dy}{dx} = \tfrac{1}{2}\left(4x-x^2\right)^{-\frac{1}{2}}\left(4-2x\right) = \tfrac{2-x}{\sqrt{4x-x^2}}$$

$$ds = \sqrt{1+\left(\tfrac{dy}{dx}\right)^2}\,dx = \sqrt{1+\tfrac{(2-x)^2}{4x-x^2}}\,dx = \sqrt{\tfrac{4x-x^2+(2-x)^2}{4x-x^2}}\,dx = \sqrt{\tfrac{4}{4x-x^2}}\,dx = \tfrac{2}{\sqrt{4x-x^2}}\,dx$$

The area is then,

$$A = \int 2\pi y\,ds = \int_1^3 2\pi\sqrt{4x-x^2}\,\tfrac{2}{\sqrt{4x-x^2}}\,dx = \int_1^3 4\pi\,dx = \boxed{8\pi}$$

6.

(a) $A = \int 2\pi y\,ds = \int_{-1}^{1} 2\pi y\sqrt{1+\left(2y\mathbf{e}^{-y^2}\right)^2}\,dy = \boxed{\int_{-1}^{1} 2\pi y\sqrt{1+4y^2\mathbf{e}^{-2y^2}}\,dy}$

(b) $A = \int 2\pi x\,ds = \int_{-1}^{1} 2\pi\mathbf{e}^{-y^2}\sqrt{1+\left(2y\mathbf{e}^{-y^2}\right)^2}\,dy = \boxed{\int_{-1}^{1} 2\pi\mathbf{e}^{-y^2}\sqrt{1+4y^2\mathbf{e}^{-2y^2}}\,dy}$

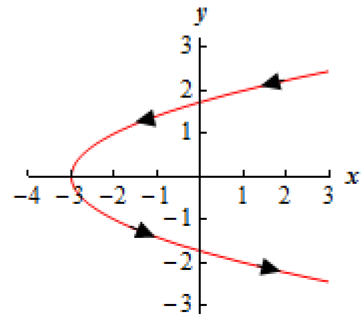
7. In this case to eliminate the parameter we'll need to solve *y* for *t* and plug that into *x*.

$$t = 2 - y \quad \rightarrow \quad x = (2-y)^2 - 4(2-y) + 1 = y^2 - 3$$

So, the graph is a parabola that opens to the right with vertex (-3, 0). A sketch is to the right. We can clearly see that $\frac{dy}{dt} = -1 < 0$ and so *y* must always be decreasing which, in turn, forces the direction shown in the sketch. Also, from the graph we can see that we have,

$$-\infty < x < \infty \qquad\qquad y \geq -3$$

The graph will trace out exactly once.



10. First eliminate the parameter.

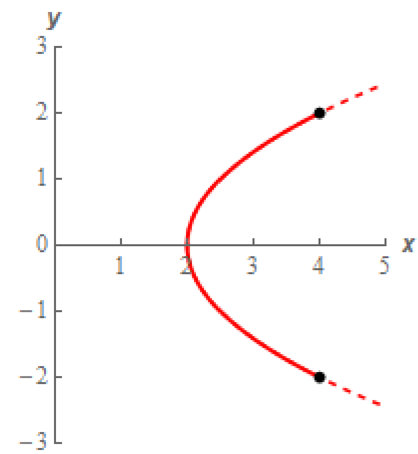
$$\cos^2(4t) = \tfrac{1}{2}(4-x) \qquad \sin(4t) = \tfrac{1}{2}y$$

$$1 = \cos^2(4t) + \sin^2(4t) = \tfrac{1}{2}(4-x) + \tfrac{1}{4}y^2 \qquad \rightarrow \qquad x = \tfrac{y^2}{4} + 2$$

So, we will have a portion of this parabola. Limits on *x* and *y* are,

$$2 \leq x \leq 4 \qquad\qquad -2 \leq y \leq 2$$

In addition, because *x* and *y* both contain sine or cosine the curve will oscillate back and forth as shown to the right. Remember it MUST stay on the solid line and so can *x*/*y* only oscillate if the curve also oscillates.



We will be at the bottom/top points at,

$$\text{bottom } (y=-2):\ \sin(4t) = -1 \quad \rightarrow \quad 4t = \tfrac{3}{2}\pi + 2\pi n \quad \rightarrow \quad t = \tfrac{3}{8}\pi + \tfrac{1}{2}\pi n$$

$$\text{top } (y=2):\ \sin(4t) = 1 \quad \rightarrow \quad 4t = \tfrac{1}{2}\pi + 2\pi n \quad \rightarrow \quad t = \tfrac{1}{8}\pi + \tfrac{1}{2}\pi n$$

So, it looks like the "easiest" range of *t*'s for one trace will be : $\frac{1}{8}\pi \leq t \leq \frac{3}{8}\pi$.

The curve will trace out

$$\frac{20\pi - (-3\pi)}{\frac{1}{4}\pi} = 92 \ \text{ times}$$