3. (2 pts) Here we can see that the terms will be positive for $n > 0$ and now all we need to do is a little Calc I to verify that the terms are decreasing (at least decreasing eventually).

$$f(x) = x\mathbf{e}^{1-2x} \qquad f'(x) = -\mathbf{e}^{1-2x}(2x-1)$$

We can see that the derivative will be negative for $x > \frac{1}{2}$ and so the function, and hence the sequence terms, will be decreasing for $n > 1$ (*i.e.* eventually) and so we can use the Integral Test on this series. I'll leave it to you to verify the integration by parts work for this integral….

$$\int_0^\infty x\mathbf{e}^{1-2x}\,dx = \lim_{t\to\infty}\int_0^t x\mathbf{e}^{1-2x}\,dx = \lim_{t\to\infty}\left(-\mathbf{e}^{1-2x}\left(\tfrac{1}{4}+\tfrac{x}{2}\right)\right)\Big|_0^t = \lim_{t\to\infty}\left(-\mathbf{e}^{1-2t}\left(\tfrac{1}{4}+\tfrac{t}{2}\right)+\tfrac{1}{4}\mathbf{e}\right) = \tfrac{1}{4}\mathbf{e}$$

The integral converges (the limit required L'Hospitals Rule which I'm assuming you can do right?) and so by the **Integral Test** the series will also **converge.**

5. (2 pts) The terms of this series are clearly (hopefully) positive so we can use the Comparison Test. It looks like the series terms will behave like $\frac{3n^2}{n^3} = \frac{3}{n}$ and so it's a pretty good guess that the series will diverge. We will then need a smaller series that we know diverges.

$$\begin{aligned}\frac{3n^2+\sin^2 n}{n^3\cos^2 n} &\ge \frac{3n^2+0}{n^3\cos^2 n} && 0\le \sin^2 n \le 1\\ &\ge \frac{3n^2+0}{n^3(1)} && 0\le\cos^2 n\le 1\\ &= \frac{3}{n}\end{aligned}$$

Now, $\displaystyle\sum_{n=1}^\infty \frac{3}{n}$ is a harmonic series and so diverges. Therefore, by the **Comparison Test** the original series will **diverge**.

6. (2 pts) The terms of this series are positive eventually (for $n > \frac{9}{5} = 1.8$) so we can use the Limit Comparison Test. This looks to be a good candidate for the Limit Comparison Test and we can use

$$\sum_{n=1}^\infty \frac{n}{n^2} = \sum_{n=1}^\infty \frac{1}{n}$$

as the 2nd series and we know that this is a harmonic series which will diverge. Note that with the "-" in both the numerator and denominator we can't use the normal Comparison Test.

$$\lim_{n\to\infty}\frac{5n-9}{7n^2-n+8}\,\frac{n}{1} = \lim_{n\to\infty}\frac{5n^2-9n}{7n^2-n+8} = \frac{5}{7}$$

So, because $0 < c = \frac{5}{7} < \infty$ we know that both series will have the same convergence and because we know the second series diverges by the **Limit Comparison Test** we also know that the original series will **diverge.**

8. (2 pts) First note that we can rewrite this as,

$$\sum_{n=0}^{\infty}\frac{1}{(-2)^{n+2}(n^2+1)}=\sum_{n=0}^{\infty}\frac{1}{(-1)^{n+2}\,2^{n+2}(n^2+1)}=\sum_{n=0}^{\infty}\frac{(-1)^{n+2}}{2^{n+2}(n^2+1)}$$

So, this is an alternating series (just written oddly) with,

$$b_n=\frac{1}{2^{n+2}(n^2+1)}>0 \qquad\qquad \lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{1}{2^{n+2}(n^2+1)}=0$$

It is (hopefully) clear that increasing *n* can only increase the denominator and so the terms are decreasing. Therefore, by the **Alternating Series Test** this series will **converge.**

10. (2 pts)

$$L=\lim_{n\to\infty}\left|\frac{(n-1)!}{(2n+3)!}\,\frac{(2n+1)!}{(n-2)!}\right|=\lim_{n\to\infty}\left|\frac{(n-1)(n-2)!}{(2n+3)(2n+2)(2n+1)!}\,\frac{(2n+1)!}{(n-2)!}\right|$$

$$=\lim_{n\to\infty}\left|\frac{n-1}{(2n+3)(2n+2)}\right|=0<1$$

So, by the **Ratio Test** this series will **converge.**

Not Graded

1. In this case all we need to do is rewrite the series a little and the use the ***p*-series test**.

$$\sum_{n=8}^{\infty}\frac{3}{n^{\frac{1}{3}}\,n^{\frac{4}{7}}}=\sum_{n=8}^{\infty}\frac{3}{n^{\frac{19}{21}}}$$

The exponent is then $\frac{19}{21}<1$ and so by the ***p*-series test** this series will **diverge**.

2. In this case we can see that the series term will be positive for $n\geq 2$ and increasing *n* will only increase the denominator of the series terms and so the series terms are clearly decreasing. Therefore we can use the integral test on this series.

$$\int_2^{\infty}\frac{1}{x^2-1}\,dx=\lim_{t\to\infty}\int_2^t\frac{\frac{1}{2}}{x-1}-\frac{\frac{1}{2}}{x+1}\,dx=\lim_{t\to\infty}\tfrac{1}{2}\left(\ln|x-1|-\ln|x+1|\right)\Big|_2^t$$

$$=\lim_{t\to\infty}\tfrac{1}{2}\left(\ln(t-1)-\ln(t+1)-\ln(3)\right)\lim_{t\to\infty}\tfrac{1}{2}\left(\ln\left(\frac{t-1}{t+1}\right)-\ln(3)\right)$$

$$=\tfrac{1}{2}\left(\ln(1)-\ln(3)\right)=\tfrac{1}{2}\ln(3)$$

I'll leave it to you to verify my partial fraction work and we can see that the integral will converge and so by the **Integral Test** this series will **converge**.

4. The terms of this series are clearly (hopefully) positive so we can use the Comparison Test. Because of the 6^n in the denominator it a good guess that this series will converge and so we'll need a larger series that we know converges.

$$\frac{3\sin^2(2n)}{n+8^n} \le \frac{3(1)}{n+8^n} \qquad\qquad 0 \le \sin^2(2n) \le 1$$

$$\le \frac{3}{8^n} \qquad\qquad 8^n \le n+8^n$$

Now, $\sum_{n=0}^{\infty}\frac{3}{8^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{8}\right)^n$ is a geometric series with $|r| = \left|\frac{1}{8}\right| = \frac{1}{8} < 1$ and so will converge. Therefore, by the **Comparison Test** the original series will **converge**.

7. First, note that because $\cos(n\pi) = (-1)^n$ this really is an alternating series with,

$$b_n = \frac{1}{n^2+1} > 0 \qquad\qquad \lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n^2+1} = 0$$

Also, because increasing *n* only increases the denominator these will decrease and so by the **Alternating Series Test** this series will **converge.**

9. This is an alternating series with,

$$b_n = \frac{n}{n^2+n+4} > 0 \qquad\qquad \lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{n^2+n+4} = 0$$

It is not clear that these are decreasing and so we'll need to do some Calc I to prove that.

$$f(x) = \frac{x}{x^2+x+4} \qquad\qquad f'(x) = \frac{4-x^2}{\left(x^2+x+4\right)^2} \qquad \rightarrow \qquad x = \pm 2$$

I'll leave it to you to verify the two critical points shown above and that provided $x > 2$ the derivative is negative and the function is decreasing. Hence the terms are decreasing eventually and so by the **Alternating Series Test** this series **converges**.

11. $L = \lim_{n\to\infty}\left|\frac{(-5)^{n+3}(n+1)}{2^{4+3n}}\,\frac{2^{1+3n}}{(-5)^{n+2}\,n}\right| = \lim_{n\to\infty}\left|\frac{(-5)^1(n+1)}{2^3\ n}\right| = \frac{5}{8}\lim_{n\to\infty}\frac{n+1}{n} = \frac{5}{8} < 1$

By the **Ratio Test** this series will **converge**.

12. $L = \lim_{n\to\infty}\left|\frac{(n+1)^2\,3^{-1-2n}}{n^2\,3^{1-2n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2\,3^2}\right| = \frac{1}{9}\lim_{n\to\infty}\frac{(n+1)^2}{n^2} = \frac{1}{9} < 1$

By the **Ratio Test** this series will **converge**.

13. $L = \lim_{n\to\infty} \left| \left(\left(\frac{13+6n^3}{7n+2n^3} \right)^{2n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \left(\frac{13+6n^3}{7n+2n^3} \right)^{2} \right| = \left| \left(\frac{6}{2} \right)^{2} \right| = 9 > 1$

So, by the **Root Test** this series will **diverge.**

14. Do not make the mistake of saying that this series is a harmonic series and hence diverges. Because there are two of them here it is not a harmonic series. This is a telescoping series so we'll need to set up the partial sums.

$$s_N = \sum_{n=2}^{N} \frac{1}{n} - \frac{1}{n-1} = \left(\cancel{\frac{1}{2}} - \frac{1}{1} \right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{2}} \right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{3}} \right) + \cdots$$

$$\cdots + \left(\cancel{\frac{1}{N-2}} - \cancel{\frac{1}{N-3}} \right) + \left(\cancel{\frac{1}{N-1}} - \cancel{\frac{1}{N-2}} \right) + \left(\frac{1}{N} - \cancel{\frac{1}{N-1}} \right)$$

$$= -1 + \frac{1}{N} \qquad \lim_{n\to\infty} s_N = \lim_{n\to\infty} \left(-1 + \frac{1}{N} \right) = -1$$

The sequence of partial sums converges and so the series **converges** (and has a value of -1).

15. Note that this is an alternating series as the alternating sign can be in either the numerator or denominator. We can always move it to the numerator if we want as follows.

$$\sum_{n=1}^{\infty} \frac{4}{(-1)^{n+2}(n+3)} \frac{(-1)^{n+2}}{(-1)^{n+2}} \sum_{n=1}^{\infty} \frac{4(-1)^{n+2}}{n+3}$$

So,

$$b_n = \frac{4}{n+3} > 0 \qquad \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{4}{n+3} = 0$$

Increasing *n* only increases the denominator and so these will decrease and so by the **Alternating Series Test** this series will **converge**.

16. $L = \lim_{n\to\infty} \left| \frac{(n+1)^2}{3^{-1-2n}(-4)^{5+n}} \frac{3^{1-2n}(-4)^{4+n}}{n^2} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{(-4)^1} \frac{3^2}{n^2} \right| = \frac{9}{4} \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \frac{9}{4} > 1$

17. These terms are positive and if we do some Calc I we get,

$$f(x) = \frac{[\ln x]^2}{x} \qquad f'(x) = \frac{2\ln x - [\ln x]^2}{x^2} = \frac{\ln x(2 - \ln x)}{x^2} \quad \to \quad \ln x = 2 \quad \to \quad x = \mathbf{e}^2$$

This function has a single critical point as shown above and I'll leave it to you to verify that the derivative will be negative for $x > \mathbf{e}^2$ and so the function will be eventually be decreasing and so we can use the Integral Test on this series.

$$\int_1^{\infty} \frac{[\ln x]^2}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{[\ln x]^2}{x}\,dx = \lim_{t\to\infty} \tfrac{1}{3}[\ln x]^3\Big|_1^t = \lim_{t\to\infty} \tfrac{1}{3}[\ln t]^3 = \infty$$

So, this integral is divergent and so by the **Integral Test** the series **diverges**.

18.

$$L = \lim_{n\to\infty}\left|\frac{2^{n+2}}{(2n+3)!}\frac{(2n+1)!}{2^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2^{n+2}}{(2n+3)(2n+2)(2n+1)!}\frac{(2n+1)!}{2^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2^1}{(2n+3)(2n+2)}\right| = 0 < 1$$

So, by the **Ratio Test** this series will **converge.**