**3. (3 pts)** $L = \lim_{n\to\infty} \left| \frac{(2x+6)^{n+1}}{(-3)^n} \frac{(-3)^{n-1}}{(2x+6)^n} \right| = \lim_{n\to\infty} \left| \frac{(2x+6)}{(-3)} \right| = \frac{1}{3}|2x+6| = \frac{2}{3}|x+3|$

$$\tfrac{2}{3}|x+3| < 1 \qquad |x+3| < \tfrac{3}{2} \qquad -\tfrac{3}{2} < x+3 < \tfrac{3}{2} \qquad \to \qquad -\tfrac{9}{2} < x < -\tfrac{3}{2}$$

Checking the endpoints gives,

$x = -\frac{9}{2}: \sum_{n=0}^{\infty} \frac{(-3)^n}{(-3)^{n-1}} = \sum_{n=0}^{\infty} (-3)$ Diverges - Divergence Test

$x = -\frac{3}{2}: \sum_{n=0}^{\infty} \frac{(3)^n}{(-3)^{n-1}} = \sum_{n=0}^{\infty} \frac{(3)^n}{(-1)^{n-1}(3)^{n-1}} = \sum_{n=0}^{\infty} \frac{3}{(-1)^{n+1}}$ Diverges - Divergence Test

The **Radius of Convergence** is $R = 2$ and the **Interval of Convergence** is $-\frac{9}{2} < x < -\frac{3}{2}$.

**5. (2 pts)**

$$g(t) = \frac{4t}{7} \frac{1}{1-\frac{1}{7}\sqrt{t}} = \frac{4t}{7} \sum_{n=0}^{\infty} \left( \tfrac{1}{7} t^{\frac{1}{2}} \right)^n = \frac{4t}{7} \sum_{n=0}^{\infty} \tfrac{1}{7^n} t^{\frac{1}{2}n} = \boxed{\sum_{n=0}^{\infty} \tfrac{4}{7^{n+1}} t^{\frac{1}{2}n+1}}$$

This series will converge for,

$$\left| \tfrac{1}{7} t^{\frac{1}{2}} \right| < 1 \quad \to \quad \tfrac{1}{7}|t|^{\frac{1}{2}} < 1 \quad \to \quad |t|^{\frac{1}{2}} < 7 \quad \to \quad |t| < 49$$

**7. (2 pts)** This can be done with the formula we derived in class.

$$f(x) = x^2 \cos\left(4\sqrt{x}\right) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n \left(4x^{\frac{1}{2}}\right)^{2n}}{(2n)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (16^n) x^{n+2}}{(2n)!}}$$

**9. (3 pts)** First get the derivatives and their value at the point.

$f(x) = \ln(7-5x)$ $\qquad f(0) = \ln(7)$

$f'(x) = \frac{-5}{7-5x} = -3(7-5x)^{-1}$

$f''(x) = -(5^2)(7-5x)^{-2}$

$f'''(x) = -(5^3)(2)(7-5x)^{-3}$

$f^{(4)}(x) = -(5^4)(2)(3)(7-5x)^{-4}$

$f^{(5)}(x) = -(5^5)(2)(3)(4)(7-5x)^{-5}$

$\vdots$

$f^{(n)}(x) = \frac{-(5^n)(n-1)!}{(7-5x)^n}, \quad n \ge 1$ $\qquad f^{(n)}(0) = \frac{-(5^n)(n-1)!}{(7)^n} = -\left(\frac{5}{7}\right)^n (n-1)!, \quad n \ge 1$

Note that the general formula is NOT valid for $n=0$. The Taylor series is then,

$$\ln(7-5x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=f(0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$
$$=\ln(7)+\sum_{n=1}^{\infty}-\left(\frac{5}{7}\right)^n\frac{(n-1)!}{n!}x^n=\boxed{\ln(7)+\sum_{n=1}^{\infty}-\left(\frac{5}{7}\right)^n\left(\frac{1}{n}\right)x^n}$$

---

***Not Graded***

**1.** $L=\lim_{n\to\infty}\left|\frac{(4x-1)^n}{(2n)^n}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\frac{(4x-1)}{(2n)}\right|=|4x+1|\lim_{n\to\infty}\frac{1}{2n}=0$

So, because $L<1$ for all *x*'s this series will converge for all *x* and so the **Radius of Convergence** is $R=\infty$ and the **Interval of Convergence** is $-\infty<x<\infty$.

**2.**

$$L=\lim_{n\to\infty}\left|\frac{(n-2)!(x+8)^{n+1}}{(n-3)!(x+8)^n}\right|=\lim_{n\to\infty}\left|\frac{(n-2)(n-3)!(x+8)}{(n-3)!}\right|$$
$$=|x+8|\lim_{n\to\infty}(n-2)=\infty,\quad x\neq-8$$

So, $L=\infty$ provided $x\neq-8$ and so will only converge for $x=-8$. Therefore, the **Radius of Convergence** is $R=0$ and the **Interval of Convergence** is $x=-8$.

**4.** $L=\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{2^{3n+4}(n+2)}\frac{2^{3n+1}(n+1)}{(x-1)^n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)(x-1)}{2^3(n+2)}\right|=\frac{1}{8}|x-1|\lim_{n\to\infty}\frac{n+1}{n+2}=\frac{1}{8}|x-1|$

$$\frac{1}{8}|x-1|<1\qquad |x-1|<8\qquad -8<x-1<8\qquad -7<x<9$$

Checking the endpoints gives,

$$x=-7:\sum_{n=0}^{\infty}\frac{(-8)^n}{2^{3n+1}(n+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n(8)^n}{(2^3)^n(2)(n+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n(8)^n}{8^n(2)(n+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2(n+1)}$$

Convergence - Alternating Harmonic

$$x=9:\sum_{n=0}^{\infty}\frac{(8)^n}{2^{3n+1}(n+1)}=\sum_{n=0}^{\infty}\frac{(8)^n}{(2^3)^n(2)(n+1)}=\sum_{n=0}^{\infty}\frac{(8)^n}{8^n(2)(n+1)}=\sum_{n=0}^{\infty}\frac{1}{2(n+1)}$$

Divergence - Comparison Test

The **Radius of Convergence** is $R=8$ and the **Interval of Convergence** is $-7\le x<9$.

**6.**

$$f\left(x\right)=x^2\frac{1}{1+3x^4}=x^2\sum_{n=0}^{\infty}\left(-3x^4\right)^n=x^2\sum_{n=0}^{\infty}\left(-3\right)^n x^{4n}=\boxed{\sum_{n=0}^{\infty}\left(-3\right)^n x^{4n+2}}$$

This series will converge for,

$$\left|-3x^4\right|<1 \quad\rightarrow\quad 3\left|x\right|^4<1 \quad\rightarrow\quad \left|x\right|^4<\tfrac{1}{3} \quad\rightarrow\quad \left|x\right|<\tfrac{1}{\sqrt[4]{3}}$$

**8.** First get some derivatives and their value at $x=-7$ .

$$\begin{aligned} g\left(x\right)&=3x^2+18x-4 & g\left(-7\right)&=17\\ g'\left(x\right)&=6x+18 & g'\left(-7\right)&=-24\\ g''\left(x\right)&=6 & g''\left(-7\right)&=6\\ g^{(n)}\left(x\right)&=0\quad n\geq 3 & g^{(n)}\left(-7\right)&=0\quad n\geq 3 \end{aligned}$$

The Taylor Series is then,

$$\begin{aligned} 3x^2+18x-4&=\sum_{n=0}^{\infty}\frac{g^{(n)}\left(-7\right)}{n!}\left(x+7\right)^n=\frac{g^{(0)}\left(-7\right)}{0!}+\frac{g^{(1)}\left(-7\right)}{1!}\left(x+7\right)+\frac{g^{(2)}\left(-7\right)}{2!}\left(x+7\right)^2+0\\ &=\boxed{17-24\left(x+7\right)+3\left(x+7\right)^2} \end{aligned}$$

Don't worry about Taylor series that stop like this, they happen sometimes. In fact, if you think about it, it makes sense in this case. You have a polynomial and you're trying to write it as a Taylor series, which in some ways is a polynomial.

**10.** First get some derivatives and their value at $x=4$ .

$$\begin{aligned} f\left(x\right)&=4\left(1+2x\right)^{-5}\\ f'\left(x\right)&=-4\left(2\right)\left(5\right)\left(1+2x\right)^{-6}\\ f''\left(x\right)&=4\left(2^2\right)\left(5\right)\left(6\right)\left(1+2x\right)^{-7}\\ f'''\left(x\right)&=-4\left(2^3\right)\left(5\right)\left(6\right)\left(7\right)\left(1+2x\right)^{-8}\\ f^{(4)}\left(x\right)&=4\left(2^4\right)\left(5\right)\left(6\right)\left(7\right)\left(8\right)\left(1+2x\right)^{-9}\\ &\vdots \end{aligned}$$

$$f^{(n)}\left(x\right)=\frac{\left(-1\right)^n\left(2^n\right)\left(n+4\right)!}{\left(2\right)\left(3\right)\left(1+2x\right)^{n+5}} \qquad\qquad f^{(n)}\left(-4\right)=\frac{\left(-1\right)^n\left(2^n\right)\left(n+4\right)!}{6\left(-7\right)^{n+5}}\qquad n=0,1,2,\ldots$$

Note that in order to get the factorial to show up I had to multiply the numerator and denominator by a (2)(3). This is something that you need to do on occasion so don't get excited about it. The Taylor Series is then,

$$\frac{4}{(1+2x)^5}=\sum_{n=0}^{\infty}\frac{f^{(n)}(-4)}{n!}(x+4)^n=\sum_{n=0}^{\infty}\frac{(-1)^n(2^n)(n+4)!}{6(-7)^{n+5}\,n!}(x+4)^n$$

$$=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n(2^{n-1})(n+4)(n+3)(n+2)(n+1)}{6(-7)^{n+5}}(x+4)^n}$$