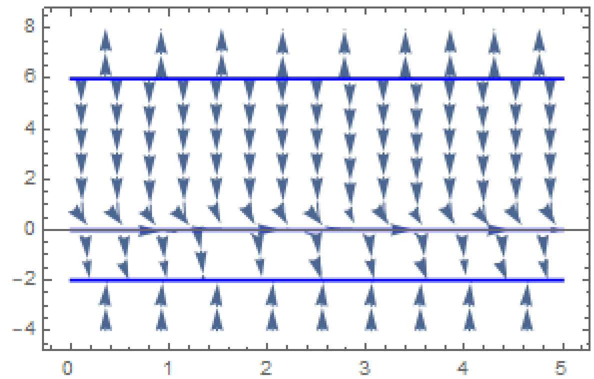
Math 3301 **Homework Set 1 – Solutions** **10 Points**

#1. (2 pts) Not much to this one. The derivative is zero at $y=-2$, $y=0$, and $y=6$. Here's a sketch of the direction field as well as a few solutions. In class we often exaggerated the curvature but the program that I used didn't and so in some cases the curvature is not very apparent. Here is the long term behavior of the solutions.



$y(0)>6$	$y(t)\to\infty$
$y(0)=6$	$y(t)=6$
$0\le y(0)<6$	$y(t)\to 0$
$y(0)<0$	$y(t)\to -2$

#3. (3 pts) Rewrite the differential equation and find the integrating factor then solve.

$$y'-\frac{4x}{x^2+2}y=\left(x^2+2\right)^2\cos(x) \qquad\Rightarrow\qquad \mu(x)=\mathbf{e}^{\int -\frac{4x}{x^2+2}dx}=\mathbf{e}^{-2\ln\left|x^2+2\right|}=\left(x^2+2\right)^{-2}$$

$$\left(x^2+2\right)^{-2}y=\int\left(\left(x^2+2\right)^{-2}y\right)'dx=\int\cos(x)\,dx=\sin(x)+c$$

$$\underline{y(x)=\left(x^2+2\right)^2\left(\sin(x)+c\right)} \qquad\Rightarrow\qquad \boxed{y(x)=\left(x^2+2\right)\left(\sin(x)-\tfrac{3}{4}\right)}$$

#5. (2 pts) This problem is NOT as difficult as it might at first seem. From Calc I we know that relative extrema (maximum in this case) occur at critical points. We also know that critical points are points where the derivative is either zero or doesn't exist. However, in this case we're told the derivative is continuous and hence will exist everywhere. Therefore, if there is a relative maximum at $t=1$ we know that $y'(1)=0$ and we're asked to determine $y(1)$. So, all we need to do is "plug" $t=1$ into the differential equation and solve as follows,

$$y'(1)-2y(1)=4-\mathbf{e}^3 \qquad\Rightarrow\qquad \boxed{y(1)=\tfrac{1}{2}\mathbf{e}^3-2=8.04277}$$

#7. (3 pts) First solve the differential equation.

$$\mu(t)=\mathbf{e}^{-5t} \qquad\Rightarrow\qquad \mathbf{e}^{-5t}y=\int 7\mathbf{e}^{-5t}-2\mathbf{e}^{-3t}\,dt=-\tfrac{7}{5}\mathbf{e}^{-5t}+\tfrac{2}{3}\mathbf{e}^{-3t}+c$$

$$\underline{y(t)=-\tfrac{7}{5}+\tfrac{2}{3}\mathbf{e}^{2t}+c\mathbf{e}^{5t}} \qquad\Rightarrow\qquad \boxed{y(t)=-\tfrac{7}{5}+\tfrac{2}{3}\mathbf{e}^{2t}+\left(\alpha^2-\tfrac{49}{15}\right)\mathbf{e}^{5t}}$$

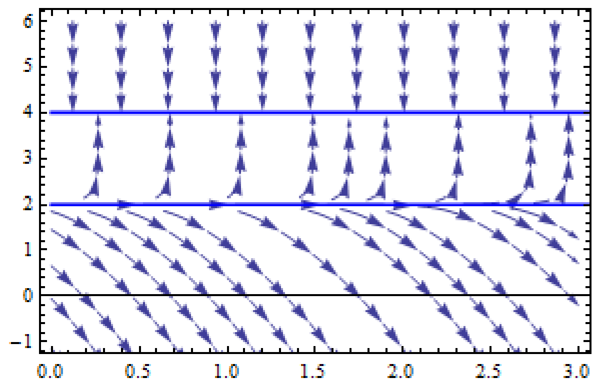
Okay, now let's work the actual problem here. Because of the exponential in the second term the first two terms will always to go ∞ as $t\to\infty$. The third term can go to 0, ∞, or $-\infty$ as $t\to\infty$ depending upon the sign of the coefficient. If the third term goes to 0 or ∞ then clearly the whole solution will go to ∞ in the long term. However, if the third term goes to $-\infty$ then because the exponent in this term is larger than that in the second term the third term will dominate and the whole solution will go to $-\infty$ in the long term. So, here is the long term behavior and its dependence upon α.

$$-\tfrac{7}{\sqrt{15}} < \alpha < \tfrac{7}{\sqrt{15}} \qquad y(t) \to -\infty \qquad\qquad \alpha \le -\tfrac{7}{\sqrt{15}},\ \ \alpha \ge \tfrac{7}{\sqrt{15}} \qquad y(t) \to \infty$$

Not Graded

#2. Not much to this one. The derivative is zero at $y = 4$ and $y = 2$.

Here's a sketch of the direction field as well as a few solutions. In class we often exaggerated the curvature of some of the solutions/arrows but the program that I used didn't and so in some cases the curvature is not very apparent. Here is the long term behavior of the solutions.



$y(0) > 2$	$y(t) \to 2$
$y(0) = 2$	$y(t) = 2$
$y(0) < 2$	$y(t) \to -\infty$

#4. Rewrite the differential equation and find the integrating factor then solve.

$$y' - \left(\tfrac{4}{t} - 6\right) y = \tfrac{1}{2} t^7 \mathbf{e}^{-6t} - \tfrac{1}{2} t^5 \mathbf{e}^{2t} \qquad \Rightarrow \qquad \mu(t) = \mathbf{e}^{-\int \frac{4}{t} - 6\, dt} = \mathbf{e}^{-4\ln|t| + 6t} = t^{-4}\mathbf{e}^{6t} \quad (\text{recall t} > 0)$$

$$t^{-4}\mathbf{e}^{6t} y = \int \left(t^{-4}\mathbf{e}^{6t} y\right)' dt = \int \tfrac{1}{2} t^3 - 8t\mathbf{e}^{8t} dt = \tfrac{1}{8} t^4 - \tfrac{1}{8}\left(8t\mathbf{e}^{8t} - \mathbf{e}^{8t}\right) + c$$

$$\underline{y(t) = \tfrac{1}{8} t^8 \mathbf{e}^{-6t} - t^5 \mathbf{e}^{2t} + \tfrac{1}{8} t^4 \mathbf{e}^{2t} + c t^4 \mathbf{e}^{-6t}} \quad \Rightarrow \quad \boxed{y(t) = \tfrac{1}{8} t^8 \mathbf{e}^{-6t} - t^5 \mathbf{e}^{2t} + \tfrac{1}{8} t^4 \mathbf{e}^{2t} + \tfrac{7}{8}\left(1 + \mathbf{e}^8\right) t^4 \mathbf{e}^{-6t}}$$

#6. First solve the differential equation.

$$\mu(t) = \mathbf{e}^{2t} \qquad \Rightarrow \qquad \mathbf{e}^{2t} y = \int \mathbf{e}^{2t} - \tfrac{3}{2} t\mathbf{e}^{2t}\, dt = \tfrac{1}{2}\mathbf{e}^{2t} + \tfrac{1}{8}\mathbf{e}^{2t}(3 - 6t) + c$$

$$\underline{y(t) = \tfrac{7}{8} - \tfrac{3}{4} t + c\mathbf{e}^{-2t}} \qquad \Rightarrow \qquad \boxed{y(t) = \tfrac{7}{8} - \tfrac{3}{4} t + \left(y_0 - \tfrac{7}{8}\right)\mathbf{e}^{-2t}}$$

Okay, now let's work the actual problem. If you think about things here it should make sense that this will only happen if there is a relative extrema (minimum or maximum) at this point, let's call it t_e. From the solution we can see that the derivative will exist everywhere and so we must have $y'(t_e) = 0$ and we'll also have $y(t_e) = 0$ since we know the solution will touch the *t*-axis here. So, plug t_e into the solution and we'll easily be able to solve for t_e.

$$2y'(t_e) + 4y(t_e) = 2 - 3t_e \qquad \Rightarrow \qquad 0 = 2 - 3t_e \qquad \Rightarrow \qquad \underline{t_e = \tfrac{2}{3}}$$

Finally, all we need to do is plug t_e into our solution above and we can solve for y_0.

$$0 = y\left(\tfrac{2}{3}\right) = \tfrac{7}{8} - \tfrac{3}{4}\left(\tfrac{2}{3}\right) + \left(y_0 - \tfrac{7}{8}\right)\mathbf{e}^{-2\left(\tfrac{2}{3}\right)} \qquad\qquad \boxed{y_0 = \tfrac{7}{8} - \tfrac{3}{8}\mathbf{e}^{\frac{4}{3}} = -0.5476}$$