Math 3301 **Homework Set 2 – Solutions** **10 Points**

#1. (3 pts) Separate, integrate, apply the initial condition and solve for the solution.

$$-\frac{1}{y}=\int\frac{1}{y^2}\,dy=\int\frac{6t}{\left(t^2-1\right)^2}\,dt=-\frac{3}{t^2-1}+c \quad\rightarrow\quad -\frac{1}{7}=3+c \quad\rightarrow\quad c=-\frac{22}{7}$$

$$-\frac{1}{y}=-\frac{3}{t^2-1}-\frac{22}{7}=\frac{1-22t^2}{7\left(t^2-1\right)} \Rightarrow \boxed{y(t)=\frac{7\left(t^2-1\right)}{1-22t^2}}$$

For the interval of validity we just need to worry about division by zero in this case so the potential intervals of validity are,

$$-\infty<t<-\sqrt{\tfrac{1}{22}} \qquad \boxed{-\sqrt{\tfrac{1}{22}}<t<\sqrt{\tfrac{1}{22}}} \qquad \sqrt{\tfrac{1}{22}}<t<\infty$$

and the second one is the actual interval because it contains $t=0$.

#2. (3 pts) Separate, integrate, apply the initial condition and solve for the solution.

$$y^2+5y=\int 2y+5\,dy=\int 3+4x\,dt=3x+2x^2+c \quad\rightarrow\quad -6=c$$

$$y^2+5y-\left(2x^2+3x-6\right)=0 \quad\rightarrow\quad y(x)=\frac{-5\pm\sqrt{25+4\left(2x^2+3x-6\right)}}{2}=\frac{-5\pm\sqrt{8x^2+12x+1}}{2}$$

Reapplying the initial condition shows that the solution is,

$$\boxed{y(x)=\frac{-5+\sqrt{8x^2+12x+1}}{2}}$$

For the interval of validity we just need to worry about square roots of negative numbers and so the potential intervals of validity are,

$$-\infty<x<\tfrac{-3-\sqrt{7}}{4} \qquad \tfrac{-3-\sqrt{7}}{4}<x<\tfrac{-3+\sqrt{7}}{4} \qquad \boxed{\tfrac{-3+\sqrt{7}}{4}<x<\infty}$$

and the third one is the actual interval because it contains $x=0$.

#5. (4 pts) Here are the two IVP's for this situation.

$$Q_1'=8(3)-(5)\frac{Q_1}{400-2t}=24-\frac{5}{2}\frac{Q_1}{200-t} \qquad Q_1(0)=10$$

$$Q_2'=(6)(4)-(2)\frac{Q_2}{200+2(t-100)}=24-\frac{Q_2}{t} \qquad Q_2(100)=q_{100}=Q_1(100)$$

Note that after 20 hours of operation there will now be 200 gallons of water in the tank and so the volume in the second IVP will need to start at 200. Also, remember that the *t*'s in the second IVP need to be shifted to account for the fact that this isn't starting at $t=0$.

Here's a quick solution for the first IVP.

$$\mu(t) = \mathbf{e}^{\int \frac{5}{2}\frac{1}{100-t}dt} = (200-t)^{-\frac{5}{2}}$$

$$\int\left((200-t)^{-\frac{5}{2}}Q_1\right)' dt = \int 24(200-t)^{-\frac{5}{2}}dt \quad \Rightarrow \quad \underline{Q_1(t) = 16(200-t) + c(200-t)^{\frac{5}{2}}}$$

$$10 = Q_1(0) = 16(200) + c(200)^{\frac{5}{2}} \Rightarrow c = -\frac{3190}{(200)^{\frac{5}{2}}} = -0.005639$$

$$\boxed{Q_1(t) = 16(200-t) - \frac{3190}{(200)^{\frac{5}{2}}}(200-t)^{\frac{5}{2}}}$$

Now, $q_{100} = Q_1(100) = 1036.0823$. To be complete the 2nd IVP is then,

$$Q_2' = 24 - \frac{Q_2}{t} \qquad Q_2(100) = 1036.0823$$

The solution to this is significantly easier than the first IVP so I'll leave it to you to verify that,

$$Q_2(t) = 12t - \frac{16391.77}{t}$$

Finally, we need to determine when this reaches 3000. So,

$$3000 = 12t - \frac{16391.77}{t} \qquad \Rightarrow \qquad t = -5.3495, \quad 255.3495$$

Now, the negative value doesn't make any sense here and so we can ignore it. So, 255.3495 hours (or 155.3495 hours after the 2nd phases started) we reach 3000 ounces.

#3. First get the solution.

$$\ln y = \int \tfrac{1}{y}dy = \int\left(4t - t^2\right)dt = 2t^2 - \tfrac{1}{3}t^3 + c \qquad \rightarrow \qquad 3 = c \qquad \Rightarrow \qquad \boxed{y(t) = \mathbf{e}^{2t^2 - \frac{1}{3}t^3 + 3}}$$

Now, for the maximum value (assuming $t > 0$, you can see why this is important correct?) Since $\lim_{t\to\infty} y(t) = 0$ we can see that the maximum value will need to occur at a critical point and from the differential equation we see that critical points will occur if,

$$0 = y\left(4t - t^2\right) \qquad \Rightarrow \qquad t = 0, 4 \qquad (y > 0 \text{ from the solution....})$$

All we need to do then is plug in the two critical points and see which is the larger as this will then be the maximum value.

$$y(0) = \mathbf{e}^3 \qquad \boxed{y(4) = \mathbf{e}^{\frac{41}{3}}}$$

<u>#4.</u> Here is the IVP for this problem.

$$Q' = 2\left(\tfrac{t}{10}\mathbf{e}^{\frac{1-t}{30}}\right) - \frac{2Q}{400} = \tfrac{t}{5}\mathbf{e}^{\frac{1-t}{30}} - \frac{Q}{200} \qquad\qquad Q(0) = 20$$

Here is the solution.

$$\mu(t) = \mathbf{e}^{\frac{1}{200}t} \qquad \to \qquad \mathbf{e}^{\frac{1}{200}t}Q = \int \tfrac{t}{5}\mathbf{e}^{\frac{1-t}{30}}\mathbf{e}^{\frac{1}{200}t}\,dt = \int \tfrac{t}{5}\mathbf{e}^{\frac{1}{30}-\frac{17}{600}t}\,dt = -\tfrac{120}{289}\mathbf{e}^{\frac{1}{30}-\frac{17}{600}t}\left(600+17t\right)+c$$

$$Q(t) = -\tfrac{120}{289}\mathbf{e}^{\frac{1-t}{30}}\left(600+17t\right) + c\mathbf{e}^{-\frac{1}{200}t} \qquad \to \qquad 20 = -\tfrac{120}{289}\mathbf{e}^{\frac{1}{30}}\left(600\right)+c \quad \to \quad c = 257.5794$$

$$\boxed{Q(t) = -\tfrac{120}{289}\mathbf{e}^{\frac{1-t}{30}}\left(600+17t\right) + 257.5794\mathbf{e}^{-\frac{1}{200}t}}$$

Now to answer the actual question : $Q(60) = 111.515$.

<u>#6.</u> Not much to this one. The volume of water in the tank when this new phase takes over is,

$$V = 200 + 2\left(255.3495 - 100\right) = 510.699$$

The new IVP is then,

$$Q_3' = (6)(4) - (4)\frac{Q_3}{510.699} = 24 - \frac{Q_3}{127.6748} \qquad Q_3\left(255.3495\right) = 3000$$