#2. (2 pts)

$$W = \begin{vmatrix} x^2 & g \\ 2x & g' \end{vmatrix} = x^2 g' - 2xg = x^4 \cos(3x)$$

So, we need to solve the following first order linear d.e.

$$x^2 g' - 2xg = x^4 \cos(3x)$$

This is a simple first order linear differential equation so I'll leave the details to you. Upon solving we get the most general possible $g(x)$,

$$\boxed{g(x) = \tfrac{1}{3} x^2 \sin(3x) + cx^2}$$

#4. (2 pts) First the complimentary solution is (I'll leave the details to you to check).

$$y_c(t) = c_1 \mathbf{e}^{10t} + c_2 t\,\mathbf{e}^{10t}$$

The guess for the particular solution (and its derivatives) is,

$$Y_p(t) = A\cos(2t) + B\sin(2t) \qquad Y_p'(t) = -2A\sin(2t) + 2B\cos(2t)$$

$$Y_p''(t) = -4A\cos(2t) - 4B\sin(2t)$$

Plugging into the differential equation and simplifying gives,

$$(96A - 40B)\cos(2t) + (40A + 96B)\sin(2t) = 13\cos(2t)$$

Setting coefficient equal and solving gives,

$$\begin{aligned} \cos(2t):&\quad 96A - 40B = 13 \\ \sin(2t):&\quad 40A - 96B = 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= \tfrac{3}{26} \\ B &= -\tfrac{5}{104} \end{aligned}$$

The general solution is then,

$$\boxed{y(t) = c_1 \mathbf{e}^{10t} + c_2 t\,\mathbf{e}^{10t} + \tfrac{3}{26}\cos(2t) - \tfrac{5}{104}\sin(2t)}$$

#6. (3 pts) First the complimentary solution is (I'll leave the details to you to check).

$$y_c(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

The guess for the particular solution (and its derivatives) is,

$$Y_p(t) = (At + B)\cos(3t) + (Dt + E)\sin(3t)$$

$$Y_p'(t) = (3Dt + A + 3E)\cos(3t) + (-3At - 3B + D)\sin(3t)$$

$$Y_p''(t) = (-9At - 9B + 6D)\cos(3t) + (-9Dt - 6A - 9E)\sin(3t)$$

Plugging into the differential equation and simplifying gives,

$$-5At\cos(3t) - 5Dt\sin(3t) + (-5B + 6D)\cos(3t) + (-6A - 5E)\sin(3t)$$

Setting coefficient equal and solving gives,

$$\begin{aligned} t\cos(3t):&\quad -5A = 0 \\ t\sin(3t):&\quad -5D = 50 \\ \cos(3t):&\quad -5B + 6D = 0 \\ \sin(3t):&\quad -6A - 5E = -5 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= 0 \\ B &= -12 \\ D &= -10 \\ E &= 1 \end{aligned}$$

The general solution is then,

$$\boxed{y(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{3t} - 12\cos(3t) + (-10t+1)\sin(3t)}$$

<u>**#7.**</u> **(3 pts)** First the complimentary solution is (I'll leave the details to you to check).

$$y_c(t) = c_1\mathbf{e}^{3t} + c_2\mathbf{e}^{6t}$$

The guess for the particular solution (and its derivatives) is,

$$Y_p(t) = At^3 + Bt^2 + Ct + D \qquad Y_p'(t) = 3At^2 + 2Bt + C \qquad Y_p''(t) = 6At + 2B$$

Plugging into the differential equation and simplifying gives,

$$18At^3 + (-27A + 18B)t^2 + (6A - 18B + 18C)t + 2B - 9C + 18D = 72t^3 + 9$$

Setting coefficient equal and solving gives,

$$\begin{aligned} t^3:&\quad 18A = 72 \\ t^2:&\quad -27A + 18B = 0 \\ t^1:&\quad 6A - 18B + 18C = 0 \\ t^0:&\quad 2B - 9C + 18D = 9 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= 4 \\ B &= 6 \\ C &= \tfrac{14}{3} \\ D &= \tfrac{13}{6} \end{aligned}$$

The general solution is then,

$$\underline{y(t) = c_1\mathbf{e}^{3t} + c_2\mathbf{e}^{6t} + 4t^3 + 6t^2 + \tfrac{14}{3}t + \tfrac{13}{6}}$$

Finally, apply the initial conditions, solve for the constants and get the actual solution.

$$\begin{aligned} c_1 + c_2 + \tfrac{13}{6} &= 2 & c_1 &= \tfrac{29}{9} \\ 3c_1 + 6c_2 + \tfrac{14}{3} &= -6 & c_2 &= -\tfrac{61}{18} \end{aligned} \qquad \Rightarrow \qquad \boxed{y(t) = \tfrac{29}{9}\mathbf{e}^{3t} - \tfrac{61}{18}\mathbf{e}^{6t} + 4t^3 + 6t^2 + \tfrac{14}{3}t + \tfrac{13}{6}}$$

Not Graded

<u>**#1.**</u> Compute the Wronskian.

$$W(y_1, y_2) = \begin{vmatrix} \mathbf{e}^{r_1 t} & \mathbf{e}^{r_2 t} \\ r_1\mathbf{e}^{r_1 t} & r_2\mathbf{e}^{r_2 t} \end{vmatrix} = r_2\mathbf{e}^{(r_1+r_2)t} - r_1\mathbf{e}^{(r_1+r_2)t} = (r_2 - r_1)\mathbf{e}^{(r_1+r_2)t} \neq 0$$

So they are a fundamental set of solutions and the general solution is what I claimed it to be and the general solution is $y(t) = c_1\mathbf{e}^{r_1 t} + c_2\mathbf{e}^{r_2 t}$.

<u>**#3.**</u> First the complimentary solution is (I'll leave the details to you to check).

$$y_c(t) = c_1\mathbf{e}^{\frac{1}{3}t}\cos(4t) + c_2\mathbf{e}^{\frac{1}{3}t}\sin(4t)$$

The guess for the particular solution (and its derivatives) is,

$$Y_p(t) = A\mathbf{e}^{-3t} \qquad Y_p'(t) = -3A\mathbf{e}^{-3t} \qquad Y_p''(t) = 9A\mathbf{e}^{-3t}$$

Plugging into the differential equation and simplifying gives,

$$244A\mathbf{e}^{-3t} = 4\mathbf{e}^{-3t} \qquad \Rightarrow \qquad A = \tfrac{4}{244} = \tfrac{1}{61}$$

The general solution is then,

$$\boxed{y(t) = c_1\mathbf{e}^{\frac{1}{3}t}\cos(4t) + c_2\mathbf{e}^{\frac{1}{3}t}\sin(4t) + \tfrac{1}{61}\mathbf{e}^{-3t}}$$

#5. First the complimentary solution is (I'll leave the details to you to check).

$$y_c(t) = c_1 \mathbf{e}^{-\frac{3}{2}t} + c_2 t\, \mathbf{e}^{-\frac{3}{2}t}$$

The guess for the particular solution (and its derivatives) is,

$$Y_p(t) = \left(At^2 + Bt + C\right)\mathbf{e}^{-t} \qquad Y_p'(t) = \left(-At^2 + \left(2A - B\right)t + B - C\right)\mathbf{e}^{-t}$$

$$Y_p''(t) = \left(At^2 + \left(-4A - B\right)t + 2A - 2B + C\right)\mathbf{e}^{-t}$$

Plugging into the differential equation and simplifying gives,

$$At^2\mathbf{e}^{-t} + \left(8A + B\right)t\mathbf{e}^{-t} + \left(8A + 4B + C\right)\mathbf{e}^{-t} = 3t^2\mathbf{e}^{-t}$$

Setting coefficient equal and solving gives,

$$\begin{array}{lrcl} t^2\mathbf{e}^{-t}: & A = 3 & & A = 3 \\ t\mathbf{e}^{-t}: & 8A + B = 0 & \Rightarrow & B = -24 \\ \mathbf{e}^{-t}: & 8A + 4B + C = 0 & & C = 72 \end{array}$$

The general solution is then,

$$\boxed{y(t) = c_1 \mathbf{e}^{-\frac{3}{2}t} + c_2 t\, \mathbf{e}^{-\frac{3}{2}t} + \left(3t^2 - 24t + 72\right)\mathbf{e}^{-t}}$$