**#2. (2 pts)** First the complimentary solution is (I'll leave the details to you to check).

$$y_c(t) = c_1 + c_2\,\mathbf{e}^{-\frac{7}{3}t}$$

The guess for the particular solution (and its derivatives) is,

$$Y_p(t) = A\mathbf{e}^{-t} + t(Bt + C) = A\mathbf{e}^{-t} + Bt^2 + Ct \qquad Y_p'(t) = -A\mathbf{e}^{-t} + 2Bt + C$$

$$Y_p''(t) = A\mathbf{e}^{-t} + 2B$$

Note that we needed to add an extra *t* to the second and third terms since a constant is part of the complimentary solution. Plugging into the differential equation and simplifying gives,

$$-4A\mathbf{e}^{-t} + 14Bt + 6B - 7C = 9 - \mathbf{e}^{-t} - 14t$$

Setting coefficient equal and solving gives,

$$\begin{array}{lll} \mathbf{e}^{-t}: & -4A = -1 & A = \frac{1}{4} \\ t^1: & 14B = -14 \quad \Rightarrow & B = -1 \\ t^0: & 6B - 7C = 9 & C = \frac{15}{7} \end{array}$$

The general solution is then,

$$\boxed{y(t) = c_1 + c_2\,\mathbf{e}^{\frac{5}{2}t} + \tfrac{1}{4}\mathbf{e}^{-t} - t^2 + \tfrac{15}{7}t}$$

**#4. (2 pts)** First the complimentary solution is (I'll leave the details to you to check).

$$y_c(t) = c_1\mathbf{e}^{6t}\cos t + c_2\,\mathbf{e}^{6t}\sin t$$

The guess for the particular solution is,

$$Y_P = \left(At^4 + Bt^3 + Ct^2 + Dt + E\right)\mathbf{e}^{6t} + t\left(F\mathbf{e}^{6t}\cos t + G\mathbf{e}^{6t}\sin t\right)$$

The second term needs an extra *t* because it is exactly the complimentary solution without the *t*.

**#7. (2 pts)**First the complimentary solution is (I'll leave the details to you to check).

$$y_c(t) = c_1\mathbf{e}^{\frac{1}{2}t} + c_2\mathbf{e}^{-4t}$$

Next, $g(t)$ and the Wronskian.

$$g(t) = 2 + \tfrac{3}{2}\mathbf{e}^{2t} \qquad W = \begin{vmatrix} \mathbf{e}^{\frac{1}{2}t} & \mathbf{e}^{-4t} \\ \frac{1}{2}\mathbf{e}^{\frac{1}{2}t} & -4\mathbf{e}^{-4t} \end{vmatrix} = \underline{-\tfrac{9}{2}\mathbf{e}^{-\frac{7}{2}t}}$$

The particular solution is then,

$$\begin{aligned} Y_P &= -\mathbf{e}^{\frac{1}{2}t}\int \frac{\mathbf{e}^{-4t}\left(2 + \frac{3}{2}\mathbf{e}^{2t}\right)}{-\frac{9}{2}\mathbf{e}^{-\frac{7}{2}t}}\,dt + \mathbf{e}^{-4t}\int \frac{\mathbf{e}^{\frac{1}{2}t}\left(2 + \frac{3}{2}\mathbf{e}^{2t}\right)}{-\frac{9}{2}\mathbf{e}^{-\frac{7}{2}t}}\,dt \\ &= \tfrac{2}{9}\mathbf{e}^{\frac{1}{2}t}\int 2\mathbf{e}^{-\frac{1}{2}t} + \tfrac{3}{2}\mathbf{e}^{\frac{3}{2}t}\,dt - \tfrac{2}{9}\mathbf{e}^{-4t}\int 2\mathbf{e}^{4t} + \tfrac{3}{2}\mathbf{e}^{6t}\,dt \\ &= \tfrac{2}{9}\mathbf{e}^{\frac{1}{2}t}\left(-4\mathbf{e}^{-\frac{1}{2}t} + \mathbf{e}^{\frac{3}{2}t}\right) - \tfrac{2}{9}\mathbf{e}^{-4t}\left(\tfrac{1}{2}\mathbf{e}^{4t} + \tfrac{1}{4}\mathbf{e}^{6t}\right) = \boxed{-1 + \tfrac{1}{6}\mathbf{e}^{2t}} \end{aligned}$$

The general solution is then,

$$\underline{y(t) = c_1\mathbf{e}^{\frac{1}{2}t} + c_2\mathbf{e}^{-4t} - 1 + \tfrac{1}{6}\mathbf{e}^{2t}}$$

Finally apply the initial conditions, solve for the constants and find the actual solution.

$$\begin{aligned} c_1 + c_2 - 1 + \tfrac{1}{6} &= -2 \qquad c_1 = -\tfrac{8}{9} \\ \tfrac{1}{2}c_1 - 4c_2 + \tfrac{1}{3} &= 1 \qquad c_2 = -\tfrac{5}{18} \end{aligned} \qquad \Rightarrow \qquad \boxed{y(t) = -\tfrac{8}{9}\mathbf{e}^{\frac{1}{2}t} - \tfrac{5}{18}\mathbf{e}^{-4t} - 1 + \tfrac{1}{6}\mathbf{e}^{2t}}$$

**#9. (2 pts)** Here are the important quantities.

$$m = 0.06 \qquad k = \frac{9.8(0.06)}{0.01} = 58.8 \qquad \gamma = \frac{0.75}{0.0625} = 12 \qquad \gamma_{CR} = 2\sqrt{(58.8)(.06)} = 3.7566$$

It looks like we're overdamped in this case. The IVP is,

$$0.06u'' + 12u' + 58.8u = 0 \qquad u(0) = 0.10 \quad u'(0) = -0.08$$

The general solution to this is,

$$u(t) = c_1\mathbf{e}^{\left(-100+2\sqrt{2255}\right)t} + c_2\mathbf{e}^{\left(-100-2\sqrt{2255}\right)t} = c_1\mathbf{e}^{-5.0263t} + c_2\mathbf{e}^{-194.9737t}$$

Applying the initial conditions gives the actual solution of,

$$\boxed{u(t) = 0.1022\mathbf{e}^{-5.0263t} - 0.002225\mathbf{e}^{-194.9737t}}$$

**#11. (2 pts)** The IVP for this case is (using the previous work from **#9**).

$$0.06u'' + 12u' + 58.8u = 7\cos(2t) \qquad u(0) = 0.10 \quad u'(0) = -0.08$$

The complimentary solution from **#9** is,

$$u_c(t) = c_1\mathbf{e}^{-5.0263t} + c_2\mathbf{e}^{-194.9737t}$$

The form of the particular solution will be,

$$U_p(t) = A\cos(2t) + B\sin(2t)$$

Differentiating this, plugging into the differential equation and simplifying will give,

$$(58.56A + 24B)\cos(2t) + (-24A + 58.56B)\sin(2t) = 7\cos(2t)$$

Setting coefficients equal gives,

$$\left.\begin{aligned} \cos(2t):& \quad 58.56A + 24B = 7 \\ \sin(2t):& \quad -24A + 58.56B = 0 \end{aligned}\right\} \qquad \Rightarrow \qquad \begin{aligned} A &= 0.10235 \\ B &= 0.04194 \end{aligned}$$

So, the particular and general solutions are then,

$$U_p(t) = 0.10235\cos(2t) + 0.04194\sin(2t)$$

$$u(t) = c_1\mathbf{e}^{-5.0263t} + c_2\mathbf{e}^{-194.9737t} + 0.10235\cos(2t) + 0.04194\sin(2t)$$

Applying the initial conditions gives,

$$u(t) = -0.0032699\mathbf{e}^{-5.0263t} + 0.00092487\mathbf{e}^{-194.9737t} + 0.10235\cos(2t) + 0.04194\sin(2t)$$

The final step is to combine the last two terms into a single cosine.

$$R = \sqrt{(0.10235)^2 + (0.04194)^2} = 0.11061 \qquad \delta_1 = \tan^{-1}\left(\frac{0.04194}{0.10235}\right) = 0.389001$$

$$\delta_2 = \delta_1 + \pi = 3.53059$$

In this case we need $\delta_1$ and so the final answer is,

$$u(t) = -0.0032699\mathbf{e}^{-5.0263t} + 0.00092487\mathbf{e}^{-194.9737t} + 0.11061\cos(2t - 0.389001)$$

***Not Graded***

**#1.** First the complimentary solution is (I'll leave the details to you to check).

$$y_c(t) = c_1\mathbf{e}^{-4t}\cos(2t) + c_2\mathbf{e}^{-4t}\sin(2t)$$

The guess for the particular solution (and its derivatives) is,

$$Y_p(t) = A + B\cos(2t) + D\sin(2t) \qquad Y_p'(t) = -2B\sin(2t) + 2D\cos(2t)$$

$$Y_p''(t) = -4B\cos(2t) - 4D\sin(2t)$$

Note that we don't need an extra *t* here because the sine and cosine both have an exponential in front of them in the complimentary solution. Plugging into the differential equation and simplifying gives,

$$20A + (16B + 16D)\cos(2t) + (-16B + 16D)\sin(2t) = 5 + 4\cos(2t) - 8\sin(2t)$$

Setting coefficient equal and solving gives,

$$\begin{aligned} \cos(2t):&\quad 16B + 16D = 4 & & A = \tfrac{1}{4} \\ \sin(2t):&\quad -16B + 16D = -8 \quad \Rightarrow & & B = \tfrac{3}{8} \\ t^0:&\quad 10A = 5 & & D = -\tfrac{1}{8} \end{aligned}$$

The general solution is then,

$$\boxed{y(t) = c_1\mathbf{e}^{-4t}\cos(2t) + c_2\mathbf{e}^{-4t}\sin(2t) + \tfrac{1}{4} + \tfrac{3}{8}\cos(2t) - \tfrac{1}{8}\sin(2t)}$$

**#3.** First the complimentary solution is (I'll leave the details to you to check).

$$y_c(t) = c_1\mathbf{e}^{-4t} + c_2\,\mathbf{e}^{2t}$$

The guess for the particular solution (and its derivatives) is,

$$Y_p(t) = A\mathbf{e}^{t} + t(Bt + C)\mathbf{e}^{2t} = A\mathbf{e}^{t} + (Bt^2 + Ct)\mathbf{e}^{2t}$$

$$Y_p'(t) = A\mathbf{e}^{t} + (2Bt^2 + (2B + 2C)t + C)\mathbf{e}^{2t}$$

$$Y_p''(t) = A\mathbf{e}^{t} + (4Bt^2 + (8B + 4C)t + 2B + 4C)\mathbf{e}^{2t}$$

Note that we had to add an extra *t* onto the second term because $C\mathbf{e}^{2t}$ is part of the complimentary solution. Plugging into the differential equation and simplifying gives,

$$12Bt\mathbf{e}^{2t} + (2B + 6C)\mathbf{e}^{2t} - 5A\mathbf{e}^{-4t} = 20\mathbf{e}^{t} - 90t\mathbf{e}^{2t}$$

Setting coefficient equal and solving gives,

$$\begin{aligned} t\mathbf{e}^{2t}:&\quad 12B = -90 & & A = -4 \\ \mathbf{e}^{2t}:&\quad 2B + 6C = 0 \quad \Rightarrow & & B = -\tfrac{15}{2} \\ \mathbf{e}^{t}:&\quad -5A = 20 & & C = \tfrac{5}{2} \end{aligned}$$

The general solution is then,

$$\underline{y(t) = c_1\mathbf{e}^{-4t} + c_2\,\mathbf{e}^{2t} - 4\mathbf{e}^{t} + \tfrac{5}{2}(t - 3t^2)\mathbf{e}^{2t}}$$

Finally, apply the initial conditions, solve for the constants and get the actual solution.

$$c_1 + c_2 - 4 = 0 \qquad c_1 = \tfrac{17}{12}$$
$$-4c_1 + 2c_2 - 4 + \tfrac{5}{2} = -2 \quad c_2 = \tfrac{31}{12}$$
$$y(t) = \tfrac{17}{12}\mathbf{e}^{-4t} + \tfrac{31}{12}\mathbf{e}^{2t} - 4\mathbf{e}^{t} + \tfrac{5}{2}\left(t - 3t^2\right)\mathbf{e}^{2t}$$

**#5.** First the complimentary solution is (I'll leave the details to you to check).

$$y_c(t) = c_1\mathbf{e}^{-12t} + c_2 t\,\mathbf{e}^{-12t}$$

The guess for the particular solution is,

$$Y_P = (At + B)\cos(4t) + (Ct + D)\sin(4t) + t^2\left(Et^2 + Ft + G\right)\mathbf{e}^{-12t}$$

The second term needs an extra $t^2$ because with no *t* or a single *t* a term from the complimentary solution is buried in it.

**#6.** First the complimentary solution is (I'll leave the details to you to check).

$$y_c(t) = c_1\mathbf{e}^{-\frac{1}{2}t}\cos(2t) + c_2\mathbf{e}^{-\frac{1}{2}t}\sin(2t)$$

Next, $g(t) = 3\mathbf{e}^{-\frac{1}{2}t}$ (recall, the $y''$ needs a coefficient of 1) and the Wronskian is,

$$W = \begin{vmatrix} \mathbf{e}^{-\frac{1}{2}t}\cos(2t) & \mathbf{e}^{-\frac{1}{2}t}\sin(2t) \\ -\frac{1}{2}\mathbf{e}^{-\frac{1}{2}t}\cos(2t) - 2\mathbf{e}^{-\frac{1}{2}t}\sin(2t) & -\frac{1}{2}\mathbf{e}^{-\frac{1}{2}t}\sin(2t) + 2\mathbf{e}^{-\frac{1}{2}t}\cos(2t) \end{vmatrix}$$
$$= -\tfrac{1}{2}\mathbf{e}^{-t}\sin(2t)\cos(2t) + 2\mathbf{e}^{-t}\cos^2(2t) - \left(-\tfrac{1}{2}\mathbf{e}^{-t}\cos(2t)\sin(2t) - 2\mathbf{e}^{-t}\sin^2(2t)\right)$$
$$= 2\mathbf{e}^{-t}\cos^2(2t) + 2\mathbf{e}^{-t}\sin^2(2t) = \underline{2\mathbf{e}^{-t}}$$

The particular solution is then,

$$Y_P = -\mathbf{e}^{-\frac{1}{2}t}\cos(2t)\int \frac{\mathbf{e}^{-\frac{1}{2}t}\sin(2t)\left(3\mathbf{e}^{-\frac{1}{2}t}\right)}{2\mathbf{e}^{-t}}dt + \mathbf{e}^{-\frac{1}{2}t}\sin(2t)\int \frac{\mathbf{e}^{-\frac{1}{2}t}\cos(2t)\left(3\mathbf{e}^{-\frac{1}{2}t}\right)}{2\mathbf{e}^{-t}}dt$$
$$= -\tfrac{3}{2}\mathbf{e}^{-\frac{1}{2}t}\cos(2t)\int \sin(2t)\,dt + \tfrac{3}{2}\mathbf{e}^{-\frac{1}{2}t}\sin(2t)\int \cos(2t)\,dt$$
$$= \tfrac{3}{4}\mathbf{e}^{-\frac{1}{2}t}\cos^2(2t) + \tfrac{3}{4}\mathbf{e}^{-\frac{1}{2}t}\sin^2(2t) = \boxed{\tfrac{3}{4}\mathbf{e}^{-\frac{1}{2}t}}$$

The general solution is then,

$$\boxed{y(t) = c_1\mathbf{e}^{-\frac{1}{2}t}\cos(2t) + c_2\mathbf{e}^{-\frac{1}{2}t}\sin(2t) + \tfrac{3}{4}\mathbf{e}^{-\frac{1}{2}t}}$$

**#8.** Here are all the important quantities.

$$m = \frac{\frac{1}{4}}{32} = \frac{1}{128} \qquad k = \frac{\frac{1}{4}}{\frac{6}{12}} = \frac{1}{2} \qquad \omega_0 = \sqrt{\frac{\frac{1}{2}}{\frac{1}{128}}} = 8$$

The IVP is,

$$\tfrac{1}{128}u'' + \tfrac{1}{2}u = 0 \qquad u(0) = -\tfrac{1}{12} \quad u'(0) = -\tfrac{5}{12}$$

The general solution is

$$u(t) = c_1 \cos(8t) + c_2 \sin(8t)$$

Applying the initial conditions gives,

$$u(t) = -\tfrac{1}{12}\cos(8t) - \tfrac{5}{96}\sin(8t)$$

Now convert into a single cosine.

$$R = \sqrt{\left(-\tfrac{1}{12}\right)^2 + \left(-\tfrac{5}{96}\right)^2} = \tfrac{\sqrt{89}}{96} \qquad \delta_1 = \tan^{-1}\left(\frac{-\frac{5}{96}}{-\frac{1}{12}}\right) = 0.5586$$

$$\delta_2 = \delta_1 + \pi = 3.7002$$

In this case $\delta$ is in the third quadrant and so $\delta_2$ is the correct angle. The solution is,

$$\boxed{u(t) = \tfrac{\sqrt{89}}{96}\cos(8t - 3.7002)}$$

**#10.** Here's the IVP for this case (using the previous work from **#8**).

$$\tfrac{1}{128}u'' + \tfrac{1}{2}u = \mathbf{e}^{-4t} - \sin t \qquad u(0) = -\tfrac{1}{12} \quad u'(0) = -\tfrac{5}{12}$$

From **#8** we know that the complimentary solution is,

$$u_c(t) = c_1 \cos(8t) + c_2 \sin(8t)$$

We'll use undetermined coefficients for the particular solution. The form will be,

$$U_p(t) = A\cos t + B\sin t + C\mathbf{e}^{-4t}$$

Note that at this point we know that we WON'T have resonance. Because the frequency in the forcing function sine is not $\omega = 8$ we won't need to add a *t* onto these terms and so we won't get resonance. Differentiating $U_p(t)$, plugging into the differential equation and simplifying gives,

$$\tfrac{63}{128}A\cos(8t) + \tfrac{63}{128}B\sin(8t) + \tfrac{5}{8}C\mathbf{e}^{-4t} = \mathbf{e}^{-4t} - \sin t$$

Setting coefficients equal gives,

$$\left.\begin{array}{rl} \cos t: & \tfrac{63}{128}A = 0 \\ \sin t: & \tfrac{63}{128}B = -1 \\ \mathbf{e}^{-4t}: & \tfrac{5}{8}C = 1 \end{array}\right\} \quad \Rightarrow \quad \begin{array}{l} A = 0 \\ B = -\tfrac{128}{63} \\ C = \tfrac{8}{5} \end{array}$$

The particular and general solution is then,

$$U_p(t) = -\tfrac{128}{63}\sin t + \tfrac{8}{5}\mathbf{e}^{-4t}$$

$$u(t) = c_1\cos(8t) + c_2\sin(8t) - \tfrac{128}{63}\sin t + \tfrac{8}{5}\mathbf{e}^{-4t}$$

Applying the initial conditions gives,

$$u(t) = -1.68333\cos(8t) + 1.00188\sin(8t) - \tfrac{128}{63}\sin t + \tfrac{8}{5}\mathbf{e}^{-4t}$$

The final step is to then combine the first two terms into a single cosine.

$$R = \sqrt{(-1.68333)^2 + (1.00188)^2} = 1.9589 \qquad \delta_1 = \tan^{-1}\left(\frac{1.00188}{-1.68333}\right) = -0.5369$$

$$\delta_2 = \delta_1 + \pi = 2.6047$$

We'll need $\delta_2$ here so the final answer is,

$$\boxed{u(t) = 1.9589\cos\left(8t - 2.6047\right) - \tfrac{128}{63}\sin t + \tfrac{8}{5}\mathbf{e}^{-4t}}$$