#1. (3 pts) Take the transform of everything.

$$s^2Y(s) - sy(0) - y'(0) - 4\left(sY(s) - y(0)\right) + 20Y(s) = \frac{12}{s} - \frac{\mathbf{e}^{-4s}}{s} + \frac{8\mathbf{e}^{-6s}}{s}$$

$$\left(s^2 - 4s + 20\right)Y(s) - 3s + 11 = \frac{12 - \mathbf{e}^{-4s} + 8\mathbf{e}^{-6s}}{s}$$

$$\left(s^2 - 4s + 20\right)Y(s) = \frac{12 - \mathbf{e}^{-4s} + 8\mathbf{e}^{-6s}}{s\left(s^2 - 4s + 20\right)} + \frac{3s - 11}{s^2 - 4s + 20}$$

$$Y(s) = \left(12 - \mathbf{e}^{-4s} + 8\mathbf{e}^{-6s}\right)F(s) + G(s)$$

Now we need to find $f(t)$ and $g(t)$. I'll leave the partial fractioning details to you.

$$F(s) = \frac{1}{20}\left[\frac{1}{s} - \frac{s-4}{s^2 - 4s + 20}\right] = \frac{1}{20}\left[\frac{1}{s} - \frac{s-2-2}{(s-2)^2 + 16}\right]$$

$$= \frac{1}{20}\left[\frac{1}{s} - \frac{s-2}{(s-2)^2 + 16} + \frac{2\frac{4}{4}}{(s-2)^2 + 16}\right]$$

$$\underline{f(t) = \tfrac{1}{20}\left[1 - \mathbf{e}^{2t}\cos(4t) + \tfrac{1}{2}\mathbf{e}^{2t}\sin(4t)\right]}$$

$$G(s) = \frac{3s - 11}{s^2 - 4s + 20} = \frac{3(s-2+2) - 11}{(s-2)^2 + 16} = \frac{3(s-2)}{(s-2)^2 + 16} - \frac{5\frac{4}{4}}{(s-2)^2 + 16}$$

$$\underline{g(t) = 3\mathbf{e}^{2t}\cos(4t) - \tfrac{5}{4}\mathbf{e}^{2t}\sin(4t)}$$

The actual solution is then,

$$Y(s) = 12F(s) - \mathbf{e}^{-4s}F(s) + 8\mathbf{e}^{-6s}F(s) + G(s)$$

$$\boxed{y(t) = 12f(t) - u_4(t)f(t-4) + 8u_6(t)f(t-6) + g(t)}$$

where $f(t)$ and $g(t)$ are shown above.

#3. (2 pts) Take the transform of everything.

$$s^2Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{4}{s^2} - \frac{3\mathbf{e}^{-7s}}{s^2}$$

$$\left(s^2 - 4\right)Y(s) = \frac{4 - 3\mathbf{e}^{-7s}}{s^2}$$

$$Y(s) = \frac{4 - 3\mathbf{e}^{-7s}}{s^2(s-2)(s+2)}$$

$$Y(s) = \left(4 - 3\mathbf{e}^{-7s}\right)F(s)$$

Now we need to find $f(t)$. I'll leave the partial fractioning details to you.

$$F(s) = \frac{1}{16}\left(-\frac{4}{s^2} - \frac{1}{s+2} + \frac{1}{s-2}\right) \qquad \underline{f(t) = \frac{1}{16}\left(-4t - \mathbf{e}^{-2t} + \mathbf{e}^{2t}\right)}$$

The actual solution is then,

$$Y(s) = 4F(s) - 3\mathbf{e}^{-7s}F(s) \qquad \boxed{y(t) = 4f(t) - 3u_7(t)f(t-7)}$$

where $f(t)$ is shown above.

<u>#4.</u> (3 pts) Take the transform of everything.

$$9\left(s^2Y(s) - sy(0) - y'(0)\right) - 30\left(sY(s) - y(0)\right) + 25Y(s) = \mathbf{e}^{-2s}$$

$$\left(9s^2 - 30s + 25\right)Y(s) - 90s + 300 = \mathbf{e}^{-2s}$$

$$Y(s) = \frac{\mathbf{e}^{-2s}}{(3s-5)^2} + \frac{90s - 300}{(3s-5)^2}$$

$$Y(s) = \mathbf{e}^{-2s}F(s) + G(s)$$

Now we need to find $f(t)$ and $g(t)$. I'll leave the partial fractioning details to you.

$$F(s) = \frac{1}{9\left(s - \frac{5}{3}\right)^2} \qquad \underline{f(t) = \frac{1}{9}t\mathbf{e}^{\frac{5}{3}t}}$$

$$G(s) = \frac{30}{3s+5} - \frac{150}{(3s+5)^2} = \frac{10}{s + \frac{5}{3}} - \frac{\frac{50}{3}}{\left(s + \frac{5}{3}\right)^2} \qquad \underline{g(t) = 10\mathbf{e}^{\frac{5}{3}t} - \tfrac{50}{3}t\mathbf{e}^{\frac{5}{3}t}}$$

The actual solution is then,

$$Y(s) = \mathbf{e}^{-2s}F(s) + G(s) \qquad \boxed{y(t) = u_2(t)f(t-2) + g(t)}$$

where $f(t)$ and $g(t)$ are shown above.

<u>#8.</u> (2 pts) Take the transform of everything.

$$s^2Y(s) - sy(0) - y'(0) - 6\left(sY(s) - y(0)\right) = G(s)$$

$$\left(s^2 - 6s\right)Y(s) - 8 = G(s)$$

$$Y(s) = \frac{G(s)}{s(s-6)} + \frac{8}{s(s-6)}$$

$$Y(s) = G(s)F(s) + 8F(s)$$

Now we need to find $f(t)$. I'll let you verify the partial fractions work.

$$F(s) = \frac{A}{s} + \frac{B}{s-6} = \frac{1}{6}\left(\frac{-1}{s} + \frac{1}{s-6}\right) \qquad \underline{f(t) = \tfrac{1}{6}\left(\mathbf{e}^{6t} - 1\right)}$$

The actual solution is then,

$$\boxed{y(t) = \tfrac{1}{6}\int_0^t \left(\mathbf{e}^{6\tau} - 1\right)g(\tau - t)\,d\tau + \tfrac{4}{3}\left(\mathbf{e}^{6t} - 1\right)}$$

Not Graded

#2. Take the transform of everything.

$$2\left(s^2Y(s)-sy(0)-y'(0)\right)-\left(sY(s)-y(0)\right)-6Y(s)=\frac{3\mathbf{e}^{-6s}}{s-2}$$

$$\left(2s^2-s-6\right)Y(s)-4s+8=\frac{3\mathbf{e}^{-6s}}{s-2}$$

$$Y(s)=\frac{3\mathbf{e}^{-6s}}{(2s+3)(s-2)^2}+\frac{4s-8}{(s+7)(s-2)}$$

$$Y(s)=3\mathbf{e}^{-6s}F(s)+\frac{4}{s-2}$$

Now we need to find $f(t)$. I'll leave the partial fractioning details to you.

$$F(s)=\frac{1}{49}\left(\frac{4}{2s+3}-\frac{2}{s-2}+\frac{7}{(s-2)^2}\right) \qquad \underline{f(t)=\frac{1}{49}\left(2\mathbf{e}^{-\frac{3}{2}t}-2\mathbf{e}^{2t}+7t\mathbf{e}^{2t}\right)}$$

The actual solution is then,

$$Y(s)=3\mathbf{e}^{-6s}F(s)+\frac{4}{s-2} \qquad \boxed{y(t)=3u_6(t)f(t-6)+4\mathbf{e}^{2t}}$$

where $f(t)$ and $g(t)$ are shown above.

#5. Take the transform of everything.

$$s^2Y(s)-sy(0)-y'(0)+11\left(sY(s)-y(0)\right)+18Y(s)=\frac{6\mathbf{e}^{-s}}{s}-8\mathbf{e}^{-7s}$$

$$\left(s^2+11s+18\right)Y(s)=\frac{6\mathbf{e}^{-s}}{s}-8\mathbf{e}^{-7s}$$

$$Y(s)=\frac{6\mathbf{e}^{-s}}{s(s+9)(s+2)}-\frac{8\mathbf{e}^{-7s}}{(s+9)(s+2)}$$

$$Y(s)=6\mathbf{e}^{-s}F(s)-8\mathbf{e}^{-7s}G(s)$$

Now we need to find $f(t)$ and $g(t)$. I'll leave the partial fractioning details to you.

$$F(s)=\frac{\frac{1}{18}}{s}-\frac{\frac{1}{14}}{s+2}+\frac{\frac{1}{63}}{s+9} \qquad \underline{f(t)=\tfrac{1}{18}-\tfrac{1}{14}\mathbf{e}^{-2t}+\tfrac{1}{63}\mathbf{e}^{-9t}}$$

$$G(s)=\frac{\frac{1}{7}}{s+2}-\frac{\frac{1}{7}}{s+9} \qquad \underline{g(t)=\tfrac{1}{7}\mathbf{e}^{-2t}-\tfrac{1}{7}\mathbf{e}^{-9t}}$$

The actual solution is then,

$$Y(s)=6\mathbf{e}^{-s}F(s)-8\mathbf{e}^{-7s}G(s) \qquad \boxed{y(t)=6u_1(t)f(t-1)-8u_7(t)g(t-7)}$$

where $f(t)$ and $g(t)$ are shown above.

#6. Comparing this to the definition of a convolution integral it looks like we can define

$$g(t) = t\sin(9t) \qquad\qquad h(t) = \sin(3t)$$

and then we have,

$$F(s) = G(s)H(s) = \frac{18s}{\left(s^2+81\right)^2}\frac{3}{s^2+9} = \boxed{\frac{54s}{\left(s^2+9\right)\left(s^2+81\right)^2}}$$

#7. In this case let's define,

$$F(s) = \frac{3}{s} \qquad G(s) = \frac{8}{s^2+64} \qquad\qquad \Rightarrow \qquad f(t) = 3 \qquad g(t) = \sin(8t)$$

Now, we can shift either $f(t)$ or $g(t)$ in the convolution integral, but it would definitely be easier to shift $f(t)$ in this case. So,

$$g(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t 3\sin(8\tau)\,d\tau = \left.\left(-\frac{3}{8}\cos(8t)\right)\right|_0^t = \frac{3}{8}\left(1-\cos(8t)\right)$$