#8. (4 pts) First convert to a system.

$$x_1 = y \qquad x_1' = x_2 \qquad x_1(0) = 3$$
$$x_2 = y' \qquad x_2' = -12x_1 + 8x_2 \qquad x_2(0) = 1$$
$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -12 & 8 \end{bmatrix}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Now find the eigenvalues.

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -12 & 8-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 12 = (\lambda - 2)(\lambda - 6) = 0 \quad \Rightarrow \quad \lambda_1 = 2,\ \lambda_2 = 6$$

Now the eigenvectors.

$\lambda_1 = 2:$
$$\begin{bmatrix} -2 & 1 \\ -12 & 6 \end{bmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad -2\eta_1 + \eta_2 = 0 \quad \Rightarrow \quad \eta_2 = 2\eta_1$$
$$\vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{using } \eta_1 = 1$$

$\lambda_2 = 6:$
$$\begin{bmatrix} -6 & 1 \\ -12 & 2 \end{bmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad -6\eta_1 + \eta_2 = 0 \quad \Rightarrow \quad \eta_2 = 6\eta_1$$
$$\vec{\eta}^{(2)} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \quad \text{using } \eta_1 = 1$$

The general solution is then,

$$\vec{x}(t) = c_1\mathbf{e}^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\mathbf{e}^{6t}\begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

Next, apply the initial conditions and find the constants.

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 6 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} c_1 + c_2 &= 3 \\ 2c_1 + 6c_2 &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= \tfrac{17}{4} \\ c_2 &= -\tfrac{5}{4} \end{aligned}$$

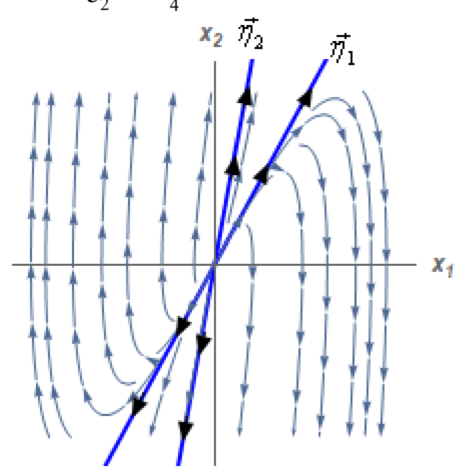
The actual solution to the system is then,

$$\vec{x}(t) = \frac{17}{4}\mathbf{e}^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{5}{4}\mathbf{e}^{6t}\begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

and the solution to the second order d.e. is,

$$\boxed{y(t) = \tfrac{17}{4}\mathbf{e}^{2t} - \tfrac{5}{4}\mathbf{e}^{6t}}$$

Now, we have two real negative eigenvalues and so we know that we'll have an **unstable** node. Here is a sketch of the phase portrait.



#9. (3 pts) First the eigenvalues.

$$\det(A-\lambda I)=\begin{vmatrix}-3-\lambda & -13\\ 1 & 3-\lambda\end{vmatrix}=\lambda^2+4=0 \quad\Rightarrow\quad \lambda_{1,2}=\pm 2i$$

Now the eigenvector for the first eigenvalue (since we only need one in this case).

$\lambda_1=2i$:

$$\begin{bmatrix}-3-2i & -13\\ 1 & 3-2i\end{bmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad\Rightarrow\quad \eta_1+(3-2i)\eta_2=0 \quad\Rightarrow\quad \eta_1=(-3+2i)\eta_2$$

$$\vec{\eta}^{(1)}=\begin{pmatrix}-3+2i\\ 1\end{pmatrix} \quad \text{using } \eta_2=1$$

Next,

$$\mathbf{e}^{2it}\begin{pmatrix}-3+2i\\ 1\end{pmatrix}=\left(\cos(2t)+i\sin(2t)\right)\begin{pmatrix}-3+2i\\ 1\end{pmatrix}$$
$$=\begin{pmatrix}-3\cos(2t)-2\sin(2t)\\ \cos(2t)\end{pmatrix}+i\begin{pmatrix}2\cos(2t)-3\sin(2t)\\ \sin(2t)\end{pmatrix}$$

The general solution is then,

$$\vec{x}(t)=c_1\begin{pmatrix}-3\cos(2t)-2\sin(2t)\\ \cos(2t)\end{pmatrix}+c_2\begin{pmatrix}2\cos(2t)-3\sin(2t)\\ \sin(2t)\end{pmatrix}$$

Apply the initial conditions and find the constants.

$$\begin{pmatrix}-2\\ 6\end{pmatrix}=\vec{x}(0)=c_1\begin{pmatrix}-3\\ 1\end{pmatrix}+c_2\begin{pmatrix}2\\ 0\end{pmatrix} \quad\Rightarrow\quad \begin{aligned}-3c_1+2c_2&=-2\\ c_1&=6\end{aligned} \quad\Rightarrow\quad \begin{aligned}c_1&=6\\ c_2&=8\end{aligned}$$

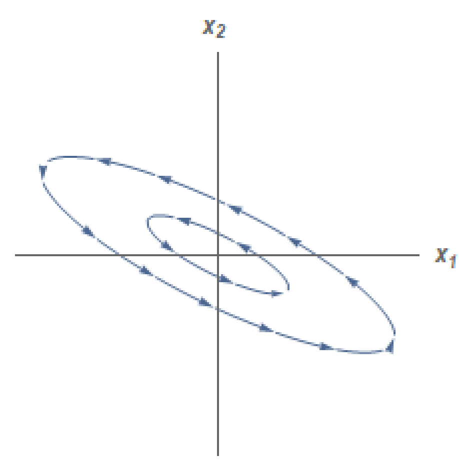
The actual solution is then,

$$\vec{x}(t)=6\begin{pmatrix}-3\cos(2t)-2\sin(2t)\\ \cos(2t)\end{pmatrix}+8\begin{pmatrix}2\cos(2t)-3\sin(2t)\\ \sin(2t)\end{pmatrix}$$
$$=\begin{pmatrix}-2\cos(2t)-36\sin(2t)\\ 6\cos(2t)+8\sin(2t)\end{pmatrix}$$

Now, we have complex eigenvalues with zero real part and so we know that we will have an **stable** center. Then using,

$$\begin{bmatrix}-3 & -13\\ 1 & 3\end{bmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}-3\\ 1\end{pmatrix}$$

we can see that as the ellipses crosses the x_1 axis they need to be going upward and so the spiral will rotate counter clock-wise. Here is a sketch.



#11. (3 pts) First the eigenvalues.

$$\det(A-\lambda I)=\begin{vmatrix}-10-\lambda & -4\\ 9 & 2-\lambda\end{vmatrix}=\lambda^2+8\lambda+16=(\lambda+4)^2=0 \quad\Rightarrow\quad \lambda_{1,2}=-4$$

Now the eigenvector (only one here…).

$$\lambda_{1,2}=-4: \qquad \begin{bmatrix}-6 & -4\\ 9 & 6\end{bmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad\Rightarrow\quad 9\eta_1+6\eta_2=0 \quad\Rightarrow\quad \eta_2=-\frac{3}{2}\eta_1$$

$$\vec{\eta}^{(1)}=\begin{pmatrix}2\\ -3\end{pmatrix} \quad \text{using } \eta_1=2$$

Next we need $\vec{\rho}$

$$\begin{bmatrix}-6 & -4\\ 9 & 6\end{bmatrix}\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix}=\begin{pmatrix}2\\ -3\end{pmatrix} \quad\Rightarrow\quad 9\rho_1+6\rho_2=-3 \quad\Rightarrow\quad \rho_2=-\tfrac{1}{2}-\tfrac{3}{2}\rho_1$$

$$\vec{\rho}=\begin{pmatrix}0\\ -\frac{1}{2}\end{pmatrix} \quad \text{using } \rho_1=0$$

The general solution is then,

$$\vec{x}(t)=c_1\mathbf{e}^{-t}\begin{pmatrix}2\\ -3\end{pmatrix}+c_2\left\{t\,\mathbf{e}^{-t}\begin{pmatrix}2\\ -3\end{pmatrix}+\mathbf{e}^{-t}\begin{pmatrix}0\\ -\frac{1}{2}\end{pmatrix}\right\}$$

Apply the initial conditions and solve for the constants.

$$\begin{pmatrix}4\\ -1\end{pmatrix}=\vec{x}(0)=c_1\begin{pmatrix}2\\ -3\end{pmatrix}+c_2\begin{pmatrix}0\\ -\frac{1}{2}\end{pmatrix} \quad\Rightarrow\quad \begin{aligned}2c_1&=4\\ -3c_1-\tfrac{1}{2}c_2&=-1\end{aligned} \quad\Rightarrow\quad \begin{aligned}c_1&=2\\ c_2&=-10\end{aligned}$$

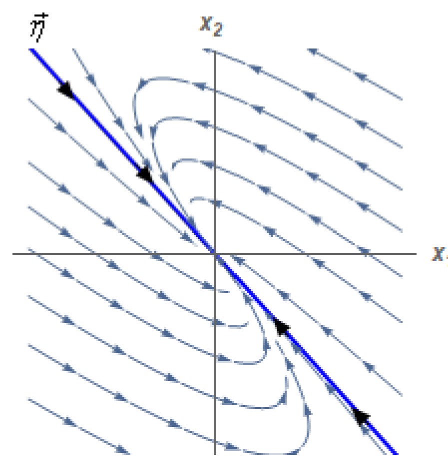
The actual solution is,

$$\vec{x}(t)=2\mathbf{e}^{-t}\begin{pmatrix}2\\ -3\end{pmatrix}-10\left\{t\,\mathbf{e}^{-t}\begin{pmatrix}2\\ -3\end{pmatrix}+\mathbf{e}^{-t}\begin{pmatrix}0\\ -\frac{1}{2}\end{pmatrix}\right\}=\mathbf{e}^{3t}\begin{pmatrix}4\\ -1\end{pmatrix}+t\,\mathbf{e}^{3t}\begin{pmatrix}-20\\ 30\end{pmatrix}$$

Now, we have a negative double eigenvalue and so we know that we will have an **asymptotically stable** improper node. Then using,

$$\begin{bmatrix}-10 & -4\\ 9 & 2\end{bmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}-10\\ 9\end{pmatrix}$$

we can see that as the trajectories cross the x_1 axis they need to be going upward. Here is a sketch.



Not Graded

#1. Here's the determinant we'll need for the eigenvalues.

$$\det\left(A-\lambda I\right)=\begin{vmatrix}10-\lambda & 1\\ -4 & 6-\lambda\end{vmatrix}=\lambda^2-16\lambda+64=\left(\lambda-8\right)^2=0\quad\Rightarrow\quad\lambda_{1,2}=8$$

So, it looks like we've got a double eigenvalue. Now, find the eigenvector.

$$\lambda_{1,2}=8:\qquad\begin{bmatrix}2 & 1\\ -4 & -2\end{bmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\quad\Rightarrow\quad 2\eta_1+\eta_2=0\quad\Rightarrow\quad\eta_2=-2\eta_1$$

$$\vec{\eta}^{(1)}=\begin{pmatrix}1\\ -2\end{pmatrix}\quad\text{using }\eta_1=1$$

#2. Here's the determinant we'll need for the eigenvalues.

$$\det\left(A-\lambda I\right)=\begin{vmatrix}\frac{3}{4}-\lambda & \frac{1}{4}\\ -37 & -\frac{1}{4}-\lambda\end{vmatrix}=\lambda^2-\tfrac{1}{2}\lambda+\tfrac{145}{16}=0\quad\Rightarrow\quad\lambda_{1,2}=\tfrac{1}{4}\pm 3i$$

Here is the eigenvector for the first eigenvalue.

$\lambda_1=\frac{1}{4}+3i$:

$$\begin{bmatrix}\frac{1}{2}-3i & \frac{1}{4}\\ -37 & -\frac{1}{2}-3i\end{bmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\quad\Rightarrow\quad\left(\tfrac{1}{2}-3i\right)\eta_1+\tfrac{1}{4}\eta_2=0\quad\Rightarrow\quad\eta_2=-4\left(\tfrac{1}{2}-3i\right)\eta_1$$

$$\vec{\eta}^{(1)}=\begin{pmatrix}1\\ -2+12i\end{pmatrix}\quad\text{using }\eta_1=1$$

Then by the fact from class the eigenvector for $\lambda_2=\frac{1}{4}-3i$ will be,

$$\vec{\eta}^{(2)}=\begin{pmatrix}1\\ -2-12i\end{pmatrix}$$

#3. Here's the determinant we'll need for the eigenvalues.

$$\det\left(A-\lambda I\right)=\begin{vmatrix}4-\lambda & 7\\ 1 & -2-\lambda\end{vmatrix}=\lambda^2-2\lambda-15=\left(\lambda-5\right)\left(\lambda+3\right)=0\quad\Rightarrow\quad\lambda_1=-3,\ \lambda_2=5$$

Here are the eigenvectors for each eigenvalue.

$$\lambda_1=-3:\qquad\begin{bmatrix}7 & 7\\ 1 & 1\end{bmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\quad\Rightarrow\quad\eta_1+\eta_2=0\quad\Rightarrow\quad\eta_1=-\eta_2$$

$$\vec{\eta}^{(1)}=\begin{pmatrix}-1\\ 1\end{pmatrix}\quad\text{using }\eta_2=1$$

$$\lambda_2=5:\qquad\begin{bmatrix}-1 & 7\\ 1 & -7\end{bmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\quad\Rightarrow\quad\eta_1-7\eta_2=0\quad\Rightarrow\quad\eta_1=7\eta_2$$

$$\vec{\eta}^{(2)}=\begin{pmatrix}7\\ 1\end{pmatrix}\quad\text{using }\eta_2=1$$

#4.

$$x_1 = y \qquad x_1' = x_2 \qquad x_1(0) = 4$$
$$x_2 = y' \qquad x_2' = -3x_1 + \tfrac{9}{7}x_2 \qquad x_2(0) = -6$$
$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -3 & \frac{9}{7} \end{bmatrix}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$$

#5.

$$x_1 = y \qquad x_1' = x_2 \qquad x_1(3) = 0$$
$$x_2 = y' \qquad x_2' = x_3 \qquad x_2(3) = -8$$
$$x_3 = y'' \qquad x_3' = x_4 \qquad x_3(3) = -5$$
$$x_4 = y''' \qquad x_4' = -x_1 + 16x_2 \qquad x_4(3) = 10$$
$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 16 & 0 & 0 \end{bmatrix}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 0 \\ -8 \\ -5 \\ 10 \end{pmatrix}$$

#6. First the eigenvalues.

$$\det(A - \lambda I) = \begin{vmatrix} -9-\lambda & 5 \\ 4 & -3-\lambda \end{vmatrix} = \lambda^2 + 12\lambda + 7 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -6 \pm \sqrt{29}$$

So, we have two real, distinct eigenvalues. Do not get excited about the fact that they are "odd". These are probably much more "realistic" than the integers and/or fractions you'll be dealing with for the rest of this homework.

Now let's find the eigenvectors for each.

$\lambda_1 = -6 + \sqrt{29}$:

$$\begin{bmatrix} -3-\sqrt{29} & 5 \\ 4 & 3-\sqrt{29} \end{bmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad -\left(3+\sqrt{29}\right)\eta_1 + 5\eta_2 = 0 \quad \Rightarrow \quad \eta_2 = \tfrac{1}{5}\left(3+\sqrt{29}\right)\eta_1$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 5 \\ 3+\sqrt{29} \end{pmatrix} \quad \text{using } \eta_1 = 5$$

$\lambda_2 = -6 - \sqrt{29}$:

$$\begin{bmatrix} -3+\sqrt{29} & 5 \\ 4 & 3+\sqrt{29} \end{bmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad -\left(3-\sqrt{29}\right)\eta_1 + 5\eta_2 = 0 \quad \Rightarrow \quad \eta_2 = \tfrac{1}{5}\left(3-\sqrt{29}\right)\eta_1$$

$$\vec{\eta}^{(2)} = \begin{pmatrix} 5 \\ 3+\sqrt{29} \end{pmatrix} \quad \text{using } \eta_1 = 5$$

The general solution is then,

$$\vec{x}(t) = c_1\mathbf{e}^{\left(-6+\sqrt{29}\right)t}\begin{pmatrix}5\\3-\sqrt{29}\end{pmatrix} + c_2\mathbf{e}^{\left(-6-\sqrt{29}\right)t}\begin{pmatrix}5\\3+\sqrt{29}\end{pmatrix}$$

#7. First the eigenvalues.

$$\det(A-\lambda I) = \begin{vmatrix}4-\lambda & 5\\ 3 & 2-\lambda\end{vmatrix} = \lambda^2 - 6\lambda - 7 = (\lambda+1)(\lambda-7) = 0 \quad \Rightarrow \quad \lambda_1 = -1,\ \lambda_2 = 7$$

Now the eigenvectors.

$\lambda_1 = -1:$
$$\begin{bmatrix}5 & 5\\ 3 & 3\end{bmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad \Rightarrow \quad 5\eta_1 + 5\eta_2 = 0 \quad \Rightarrow \quad \eta_1 = -\eta_2$$

$$\vec{\eta}^{(1)} = \begin{pmatrix}-1\\1\end{pmatrix} \quad \text{using } \eta_2 = 1$$

$\lambda_2 = 7:$
$$\begin{bmatrix}-3 & 5\\ 3 & -5\end{bmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad \Rightarrow \quad -3\eta_1 + 5\eta_2 = 0 \quad \Rightarrow \quad \eta_2 = \tfrac{3}{5}\eta_1$$

$$\vec{\eta}^{(2)} = \begin{pmatrix}5\\3\end{pmatrix} \quad \text{using } \eta_1 = 5$$

The general solution is then,

$$\vec{x}(t) = c_1\mathbf{e}^{-t}\begin{pmatrix}-1\\1\end{pmatrix} + c_2\mathbf{e}^{7t}\begin{pmatrix}5\\3\end{pmatrix}$$

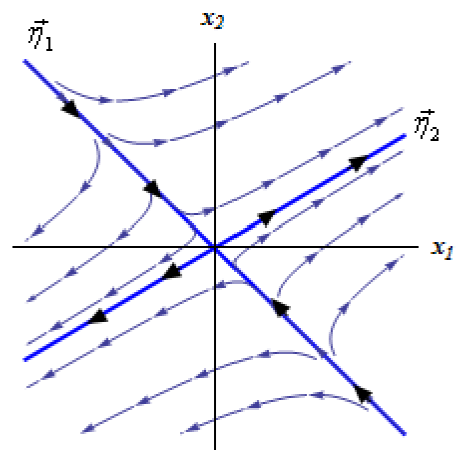
Next, apply the initial conditions and find the constants.

$$\begin{pmatrix}-8\\7\end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix}-1\\1\end{pmatrix} + c_2\begin{pmatrix}5\\3\end{pmatrix} \quad \Rightarrow \quad \begin{aligned}-c_1 + 5c_2 &= -8\\ c_1 + 3c_2 &= 7\end{aligned} \quad \Rightarrow \quad \begin{aligned}c_1 &= \tfrac{59}{8}\\ c_2 &= -\tfrac{1}{8}\end{aligned}$$

The actual solution is then,

$$\boxed{\vec{x}(t) = \frac{59}{8}\mathbf{e}^{-t}\begin{pmatrix}-1\\1\end{pmatrix} - \frac{1}{8}\mathbf{e}^{7t}\begin{pmatrix}5\\3\end{pmatrix}}$$

Now, we have two real eigenvalues with opposite signs and so we know that we'll have an **unstable** saddle point. Here is a sketch of the phase portrait.



#10. First the eigenvalues.

$$\det(A-\lambda I) = \begin{vmatrix}6-\lambda & 5\\ -10 & -4-\lambda\end{vmatrix} = \lambda^2 - 2\lambda + 26 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 1 \pm 5i$$

Now the eigenvector for the first eigenvalue (since we only need one in this case).

$\lambda_1 = 1+5i$:

$$\begin{bmatrix} 5-5i & 5 \\ -10 & -5-5i \end{bmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad (5-5i)\eta_1 + 5\eta_2 + = 0 \quad \Rightarrow \quad \eta_2 = (-1+i)\eta_1$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} \quad \text{using } \eta_1 = 1$$

Next,

$$\mathbf{e}^{(1+5i)t}\begin{pmatrix} 1 \\ -1+i \end{pmatrix} = \mathbf{e}^{t}\left(\cos(5t) + i\sin(5t)\right)\begin{pmatrix} 1 \\ -1+i \end{pmatrix}$$
$$= \mathbf{e}^{t}\begin{pmatrix} \cos(5t) \\ -\cos(5t)-\sin(5t) \end{pmatrix} + i\,\mathbf{e}^{t}\begin{pmatrix} \sin(5t) \\ \cos(5t)-\sin(5t) \end{pmatrix}$$

The general solution is then,

$$\vec{x}(t) = c_1\mathbf{e}^{t}\begin{pmatrix} \cos(5t) \\ -\cos(5t)-\sin(5t) \end{pmatrix} + c_2\,\mathbf{e}^{t}\begin{pmatrix} \sin(5t) \\ \cos(5t)-\sin(5t) \end{pmatrix}$$

Apply the initial conditions and find the constants.

$$\begin{pmatrix} -5 \\ 0 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} c_1 &= -5 \\ -c_1 + c_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= -5 \\ c_2 &= -5 \end{aligned}$$

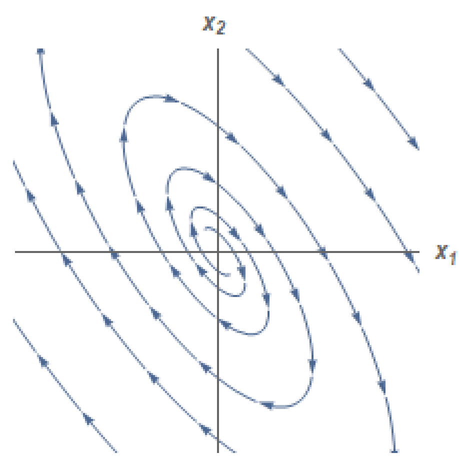
The actual solution is then,

$$\vec{x}(t) = -5\mathbf{e}^{t}\begin{pmatrix} \cos(5t) \\ -\cos(5t)-\sin(5t) \end{pmatrix} - 5\,\mathbf{e}^{t}\begin{pmatrix} \sin(5t) \\ \cos(5t)-\sin(5t) \end{pmatrix} = \mathbf{e}^{t}\begin{pmatrix} -5\cos(5t)-5\sin(5t) \\ -10\sin(5t) \end{pmatrix}$$

Now, we have complex eigenvalues with positive real part and so we know that we will have an **unstable** spiral. Then using,

$$\begin{bmatrix} 6 & 5 \\ -10 & -4 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \end{pmatrix}$$

we can see that as the arms of the spiral cross the x_1 axis they need to be going downward and so the spiral will rotate counter clock-wise. Here is a sketch.



#12. First the eigenvalues.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3}-\lambda \end{vmatrix} = \lambda^2 - \tfrac{4}{3}\lambda + \tfrac{4}{9} = \left(\lambda - \tfrac{2}{3}\right)^2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \tfrac{2}{3}$$

Now the eigenvector (only one here…).

$\lambda_{1,2} = \frac{2}{3}:$ $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \frac{1}{3}\eta_1 + \frac{1}{3}\eta_2 = 0 \Rightarrow \eta_2 = -\eta_1$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{using } \eta_1 = 1$$

Next we need $\vec{\rho}$

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \frac{1}{3}\rho_1 + \frac{1}{3}\rho_2 = 1 \Rightarrow \rho_2 = 3 - \rho_1$$

$$\vec{\rho} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad \text{using } \rho_1 = 0$$

The general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{\frac{2}{3}t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \left\{ t\,\mathbf{e}^{\frac{2}{3}t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \mathbf{e}^{\frac{2}{3}t} \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}$$

Apply the initial conditions and solve for the constants.

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 3 \end{pmatrix} \Rightarrow \begin{array}{r} c_1 = 2 \\ -c_1 + 3c_2 = 2 \end{array} \Rightarrow \begin{array}{r} c_1 = 2 \\ c_2 = \frac{4}{3} \end{array}$$

The actual solution is,

$$\vec{x}(t) = 2\mathbf{e}^{\frac{2}{3}t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{4}{3} \left\{ t\,\mathbf{e}^{\frac{2}{3}t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \mathbf{e}^{\frac{2}{3}t} \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\} = \mathbf{e}^{3t} \begin{pmatrix} 2 \\ 2 \end{pmatrix} + t\,\mathbf{e}^{3t} \begin{pmatrix} \frac{4}{3} \\ -\frac{4}{3} \end{pmatrix}$$

Now, we have a positive double eigenvalue and so we know that we will have an **unstable** improper node. Then using,

$$\begin{bmatrix} 1 & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{3} \end{pmatrix}$$

we can see that as the trajectories cross the x_1 axis they need to be going downward. Here is a sketch.