We saw that the basic ODE $y' = f(x, y)$ can be solved by Euler method. This is the lowest order method due to the fact that slope is only sampled at the start, not taking into account that slope changes before we arrive at the second point. In order to achieve higher order methods we can sample the slope at a variety of points. However this has to be done in a special way to guaranty more accuracy. Essentially we need to make sure that our sample reproduces the higher order derivative as in the Taylor series. But before we delve into the theory let's practice a few methods.
First we look at something that is reminiscent of the midpoint method in integration and for this reason we are going to call it the Mid-Point Method. Suppose we are at the point (x_n, y_n) and we want to advance to the next point (x_{n+1}, y_{n+1}). Euler method says measure the slope $k_1 = f(x_n, y_n)$ and move according to that slope, so

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ y_{n+1} &= y_n + hk_1 \end{aligned} \qquad \textit{Euler}$$

As usual $x_{n+1} = x_n + h$. Now, to improve our method, we go half way on the trajectory of Euler and pick the slope there and use that slope from the start.

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + h/2, y_n + hk_1/2) \\ y_{n+1} &= y_n + hk_2 \end{aligned} \qquad Mid - point$$

Another option is to use Heun-2 Method (by the way the names are not standard as a variety of mathematicians contributed to this area.) . Average the slope at the beginning and end of Euler method and use that slope from the start.

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + h, y_n + hk_1) \\ y_{n+1} &= y_n + h(k_1 + k_2)/2 \end{aligned} \qquad Huen - 2$$

There are many approaches which are similar to the above two formulations. One them will be referred to as the Two-thirds method (this is not an official name!).

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + 2h/3, y_n + 2hk_1/3) \\ y_{n+1} &= y_n + h(k_1 + 3k_2)/4 \end{aligned} \qquad Two - thirds$$

One of the more famous methos is referred to as the Classical Runge-Kutta Method, or $RK4$ for short. It evaluates two mid-point slopes as well as beginning and end-point slops and averages these values in a way that is reminiscent of the Simpson method in integration. Here are the details:

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + h/2, y_n + hk_1/2) \\ k_3 &= f(x_n + h/2, y_n + hk_2/2) \\ k_4 &= f(x_n + h, y_n + hk_3) \\ y_{n+1} &= y_n + h(k_1 + 2k_2 + 2k_3 + k_4)/6 \end{aligned} \qquad RK4$$

Solve a non-trivial differential equation, say $y' = xy^2$, with each of above methods.
Find the order of each method by plotting error versus step-size.