Numerical Analysis 4315/5315, Assignment 3

Topics for this lesson:
Mathematics: Approximate integration. Experimental determination of order for various quadrature rules.
Programming: Nested loops.

We want to approximate $I = \int_a^b f(x)dx$ for a smooth function over a finite interval $[a, b]$. We start by dividing $[a, b]$ into n subintervals, marked by endpoints $x_1 = a$, $x_2, x_3, \cdots, x_n$, $x_{n+1} = b$. For simplicity we take all of these to be of the same width $h = \frac{b-a}{n}$. So $x_i = a + h \cdot (i - 1)$. A Riemann sum $RS = h \cdot \Sigma_{i=1}^n f(x_i^*)$, where x_i^* is an arbitrary point in the i-th interval, $[x_i, x_{i+1}]$, gives an approximate value for the integral I. We want to discuss
(a) efficient ways of approximating I,
(b) the order of each method, determined experimentally,
(c) uses of the order-of-the-method information.

Left-endpoint rule: Here $x_i^* = x_i$, so the height of each rectangle is decided at the left endpoint of the i-th interval. We have $L_n = h \cdot \Sigma_{i=1}^n f(x_i)$.
Right-endpoint rule: Here $x_i^* = x_{i+1}$, so the height of each rectangle is decided at the right endpoint of the i-th interval. We have $R_n = h \cdot \Sigma_{i=1}^n f(x_{i+1})$.
Midpoint rule: Here $x_i^* = m_i = \frac{x_i + x_{i+1}}{2}$, so the height of each rectangle is decided at the midpoint of the i-th interval. We have $M_n = h \cdot \Sigma_{i=1}^n f(m_i)$.
Trapezoid rule: Here we approximate each panel by a trapezoid whose area is height times average of bases. Hence the sum is $T_n = h \cdot \Sigma_{i=1}^n \frac{f(x_i)+f(x_{i+1})}{2}$.

Find $\int_1^2 e^x dx$ by each of above methods and compare against the exact value for different values of n, such as 10, 100, 1000. Printout a table of actual errors, i.e. exact value of integral minus its Riemann sum, vs n for each method.
1) Notice all methods have almost same number of function calculations and similar costs (so the comparison is fair). How many function evaluation occurs in your version of the Trapezoid Rule? in Midpoint/Left/Right Rule?
2) How do the errors of various methods compare? Group the methods in terms of overall degree of accuracy.
3) How do errors of L_n and R_n compare? How can they be combined to produce a superior method?
4) How do errors of M_n and T_n compare? How can they be combined to produce a superior method? Write the corresponding summation formula for this combination method.
And more in-depth questions:
5) Why do we care about the order of a method?
6) How can this information be used to estimate or control the cost of an operation like quadratures?
7) How can the information about the order of method be used to actually improve the method?