Assignment 6

Mathematics: Newton Basins
Programming: Plotting

In your project you investigated the problem of finding the largest interval around each root of $f(x) = x^3 - 6x^2 + 11x - 6 = 0$ so that all future iterates stay within the interval and converge to the root. You might have done this by trial and error. Now we want to shed some light on this topic and find some theoretical explanation for where each interval starts and ends.
Graph the function $y = f(x)$ and try to find a geometrical significance for the end points of the interval you found by trial and error.
Try drawing the steps of Newton method on the function by hand (or better yet write a program to do it, but this may take a bit of time). It should be easy for you to find the interval around 1 and 3 by theoretical means. (Hint: You did it calculus 1, only about a 1000 times!) If you draw your graphs and tangents carefully finding the interval around 2 won't be that hard either.
Now for the bigger challenge: Can you find a (theoretical) mechanism by which you can detect other intervals, not containing 1, 2 ,or 3, but such that if you start there you will end up next to one of those roots?
The general form of this problem is referred to as Newton Basins. We consider a fixed general polynomial of degree n in the complex plane (so we use $z = a+ib$ in all calculations). This polynomial has n roots. Depending on where you start in the complex plane chances are high that you end up near one the roots after you follow several steps of Newton method. (There are usually some exceptional points, where future iterates do not come close to any root. Can you guess what else they may do?) The original point gets colored according to which root holds the key to its fate. Various shades may be used to indicate how many iterates it will take to come within certain distance of the target.
Newton basins form interesting fractal pictures. You can interactively generate them at the following site:
http://aleph0.clarku.edu/ djoyce/newton/newton.html