SIAM J. MATRIX ANAL. APPL.
Vol. 19, No. 4, pp. 1005–1026, October 1998

011

ON CONSTRUCTION OF A FAMILY OF SMOOTH NONSEPARABLE PREWAVELETS VIA INFINITE PRODUCTS OF TRIANGULARIZABLE MATRICES*

MOHSEN MAESUMI†

Abstract. Infinite products of matrices arise in many areas, such as the study of subdivision and interpolation schemes, Markov chains, and construction of wavelets of compact support. These products are used here to give sufficient conditions for the continuity and differentiability of a class of rectangular compactly supported nonseparable N-dimensional prewavelets or scaling functions. This paper considers the dilation equation $\phi(X) = \sum_K C_K \phi(2X - K)$, where $K \in \{0, \dots, m\}^N$, $\phi : \mathcal{R}^N \to \mathcal{R}$, and $C_K \in \mathcal{R}$. First, the one-dimensional case is studied, and sufficient conditions on C_K, which guarantee a continuous scaling function $\phi(X)$, are given. These conditions are based on simultaneous triangularizability of two special matrices with entries in terms of C_K. Then, these results are generalized to N dimensions and applied to the particular case where C_K's are obtained by binomial interpolation of their values at the corners of the N-cube, $\{0, m\}^N$. A set of inequalities, based on sums of C_K's on the corners of various faces of the N-cube gives sufficient conditions for the existence of smooth solutions to the dilation equation.

Key words. higher-dimensional scaling functions, infinite matrix products, simultaneous triangularizability.

AMS subject classifications. 26A15, 26A18, 41A05

PII. S0895479894262327

1. Introduction.

Infinite products of matrices occur in a wide variety of fields. They may be used to study subdivision algorithms [2, 13, 14], Markov chains [3], lattice two-scale difference equations [8, 9], and orthonormal bases of compactly supported wavelets [6].

Our interest is in characterizing certain classes of smooth compactly supported N-dimensional prewavelets or scaling functions using infinite products of matrices. These functions are the solutions of the N-dimensional dilation equations

$$\phi(X) = \sum_K C_K \phi(2X - K), \tag{1.1}$$

where $\phi : \mathcal{R}^N \to \mathcal{R}$, $C_K \in \mathcal{R}$, $K \in \{0, \dots, m\}^N$, and $X \in \mathcal{R}^N$.

If the values of ϕ at integer points are known, then one can use (1.1) to get the values of ϕ at half integers and, by iterating the process, at all dyadic points (i.e., at $\mathcal{Z}^N/2^\ell$ for all nonnegative ℓ). Since any X can be approximated by a sequence of dyadic points, e.g., through its binary expansion, the continuity of ϕ will then provide the value of $\phi(X)$.

An efficient way to describe this process is to convert (1.1) to a matrix equation. Since the support of the solution is finite, the matrix will be finite too. Then, the iteration will take the form of multiplication by certain fixed matrices with entries in terms of C_K. There is a one-to-one correspondence between the digits of the binary expansion of X and the matrices that appear in the product. As more digits of the

*Received by the editors January 28, 1994; accepted for publication (in revised form) by P. Van Dooren July 8, 1997; published electronically July 17, 1998. This research was supported in part by Texas Advanced Research Program grant 003581-005.

http://www.siam.org/journals/simax/19-4/26232.html

†Lamar University, P. O. Box 10047, Beaumont, TX, 77710 (maesumi@math.lamar.edu).

binary expansion are taken into account, the length of the matrix product increases. Hence, we will take up the question of infinite products of matrices.

In section 2 of this paper, we study the one-dimensional case. Our main result in that section, Theorem 2.5, classifies a one-parameter class of $\mathcal{C}^\ell$ scaling functions for $\ell < m-1$. In section 3, we generalize our results to the N-dimensional case. In Theorem 3.3, we classify a (2^N-1)-parameter family of $\mathcal{C}^\ell$ scaling functions in N dimensions.

2. One-dimensional scaling functions. In one dimension, the dilation equation may be written as

$$\phi(x) = c_0\phi(2x) + c_1\phi(2x-1) + \cdots + c_m\phi(2x-m), \tag{2.1}$$

where $\phi : \mathcal{R} \to \mathcal{R}$ and c_i, $i = 0, \ldots, m$, are given real coefficients. The regularity properties of the solutions of dilation equations have been extensively studied. In particular, nontrivial $\mathcal{L}^1$ solutions having compact support are characterized in [8] and shown to have their support in $[0, m]$. Moreover, it is shown that if ϕ is r times continuously differentiable, then $r < m-1$. The Hölder exponent and fractal structure of ϕ are determined in [4, 5, 9]. Continuous solutions are characterized in terms of the general and joint spectral radii of a family of matrices in [10] (see also [1, 11, 12, 15, 16]).

The point of view in the next section of this paper is to identify certain smooth one-dimensional scaling functions which lead to the specification of certain smooth solutions in higher dimensions. Some higher-dimensional scaling functions can be formed by tensor products of lower-dimensional ones. Our solution is, different however, and cannot be reduced to a tensor product.

Our construction depends on results concerning infinite products of matrices. Given a pair of matrices, T_0 and T_1, any infinite product (e.g., $P = T_0T_1T_1T_0T_1\cdots$) is associated with a binary number (e.g., $x = .01101\ldots$). We give sufficient conditions for (a) the convergence of such products; (b) the existence of a well-defined map that, given any $x \in [0,1]$, generates a product; and (c) the continuous dependence of the product on x. The sufficient conditions require that (I) the two matrices are simultaneously triangularizable by a similarity transformation, (II) the first diagonal elements of triangular matrices are 1 and the remaining elements are less than one in absolute value, and (III) the products of each matrix with the eigenvector of the other matrix (associated with eigenvalue 1) are linearly dependent. While considerably weaker conditions that guarantee the same results are known (see [10]), our requirement of simultaneous triangularizability can be easily adapted to identify certain continuous prewavelets in higher dimensions. In particular, we will characterize a (2^N-1)-parameter family of continuous $(m+1)^N$-coefficient scaling functions in N dimensions. Similar results are obtained for higher-order regularity.

2.1. Notation. Define the vector Φ and matrices T_0 and T_1 by

$$\Phi(x) = [\phi(x), \phi(x+1), \ldots, \phi(x+m-1)]^t \quad \text{for} \quad 0 \le x \le 1, \tag{2.2a}$$

$$c_k = 0 \quad \text{for} \quad k < 0 \quad \text{or} \quad k > m, \tag{2.2b}$$

$$(T_d)_{ij} = c_{2i-j+d-1} \quad \text{for} \quad 1 \le i, j \le m \quad \text{and} \quad d = 0 \text{ or } 1, \tag{2.2c}$$

$$T_0 = \begin{pmatrix} c_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c_2 & c_1 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c_4 & c_3 & c_2 & c_1 & c_0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & c_m & c_{m-1} & c_{m-2} & c_{m-3} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_m & c_{m-1} \end{pmatrix},$$

$$T_1 = \begin{pmatrix} c_1 & c_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_3 & c_2 & c_1 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_5 & c_4 & c_3 & c_2 & c_1 & c_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_m & c_{m-1} & c_{m-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_m \end{pmatrix}.$$

(In what follows, the range of i and j is $\{1, 2, \dots, m\}$ unless further restricted.) Notice that the definition of Φ is based on dividing the interval $[0, m]$ into m *cells*, $[i-1, i]$, $1 \le i \le m$. We define a pair of vectors or a function $f : \{0, 1\} \to \mathcal{R}^m$ to be *shift continuous* if $f(0)_i = f(1)_{i-1}$ for $1 < i \le m$. Obviously, $\phi(x)$ is continuous on $[0, m]$ iff Φ is continuous on $[0, 1]$ and is shift continuous.

We search for the unique normalized continuous solution ϕ with support in $[0, m]$. Sufficient conditions for the existence of such a solution and particular examples are given in the theorems of this section. This solution satisfies $\phi(x) = 0$ for $x \le 0$ or $x \ge m$, $\Phi(0)_1 = \Phi(1)_m = 0$, and

$$\Phi(x) = T_{x_1}\Phi(2x - x_1), \tag{2.3}$$

where x_1 is the first digit in the binary expansion of x. In particular if we apply (2.3) to $x = 0 = 0.00\ldots$, $x = 1 = 0.11\ldots$, and $x = 1/2 = 0.100\ldots = 0.011\ldots$, respectively, then we get

$$T_0\Phi(0) = \Phi(0), \qquad T_1\Phi(1) = \Phi(1), \tag{2.4a}$$

$$T_0\Phi(1) = T_1\Phi(0) = \Phi(1/2). \tag{2.4b}$$

Once $\Phi(0)$ or $\Phi(1)$ is known, one can calculate Φ at dyadics by repeated applications of (2.3).

Now, suppose $x \in [0, 1]$ and indicate its binary expansion by $x = 0.x_1x_2\cdots x_qx_{q+1}\cdots$. Denote by $\bar{x}_q$ the residual after the qth digit, $\bar{x}_q = 0.x_{q+1}x_{q+2}\cdots$. Then, by repeated application of (2.3), we get

$$\Phi(x) = \prod_{\ell=1}^{q} T_{x_\ell}\Phi(\bar{x}_q). \tag{2.5}$$

We define $P_q(T_0, T_1, x) = \prod_{\ell=1}^{q} T_{x_\ell}$ and $P(T_0, T_1, x) = \lim_{q\to\infty} P_q(T_0, T_1, x)$ whenever the limit exists.

Dyadic numbers have two binary expansions, e.g., $x = 1/2 = .100\ldots = .011\ldots$. Therefore, in the definition of $P_q(T_0, T_1, x)$ a particular expansion of x should be specified a priori. The consistency of (2.3) at dyadics, i.e., (2.4b), remedies this nonuniqueness for the infinite products, and the value of $P(T_0, T_1, x)$ is then determined independently of the choice of expansion for x. Further details are provided in Lemmas 2.4 and 2.5 below.

The matrices T_0 and T_1 have a very special structure. For example, the submatrices obtained by deleting the first row and column of T_0 are the same as the one obtained by deleting the last row and column of T_1. Moreover, the columns of T_0 and T_1 contain all the c_k's with an even index or all the c_k's with an odd index. However, these special properties are not used before Lemma 2.7. For this reason, and to simplify the notation, we use matrices A and B in place of T_0 and T_1, respectively.

2.2. Conditions for convergence of P_q. Some of the elementary necessary conditions for existence of $P(A, B, x)$ are expressed in the following lemma.

LEMMA 2.1. *Let Q be a finite product of A's and B's and λ be an eigenvalue of Q. Then, $P(A, B, x)$ exists for all $0 \le x \le 1$ only if $|\lambda| < 1$ or $\lambda = 1$ and nondefective.*

Proof. These conditions follow immediately from the Jordan normal form of the matrix under consideration. Here, we give a brief indication. If an eigenvalue $|\lambda| > 1$, then Q^ν is exponentially unbounded as $\nu \to \infty$, and the corresponding product does not exist. If $\lambda = 1$ is defective (i.e., its geometric multiplicity is less than its algebraic multiplicity), then Q^ν is polynomially unbounded. If $|\lambda| = 1$ but $\lambda \ne 1$, then Q^ν does not have a limit. □

We note that if all the eigenvalues of Q are less than 1 in absolute value, then $P(A, B, x)$ will be zero on a dense set of values of x. (To see this, consider the set of numbers whose binary expansions end in an infinite repetition of the digit pattern associated with Q. These numbers form a dense set and P is zero on this set.) One of the simplest cases for controlling the eigenvalues of products of matrices is when the matrices are triangular. This prompts the following definition.

DEFINITION 2.2. *A finite family of matrices $\{\mathcal{A}\}$ is said to be jointly tied (to 1) if the matrices are simultaneously lower triangularizable by a similarity transformation, their leading eigenvalue is 1, and the remaining eigenvalues are less than 1 in absolute value. Hence, there is an invertible matrix S such that for each $A \in \mathcal{A}$ we have*

$$S^{-1}AS = \tilde{A}, \quad \tilde{A}_{ij} = 0 \quad \textit{for} \quad j > i,$$

$$\tilde{A}_{11} = 1, \quad \textit{and} \quad \max_{i>1} |\tilde{A}_{ii}| < 1.$$

The following definition will be used to study the relationship between the two products associated with the two expansions of the dyadics.

DEFINITION 2.3. *Two matrices A and B are called consistent (with respect to a simple joint eigenvalue λ) if there are V_A and V_B, eigenvectors of A and B associated with λ, such that $AV_B = BV_A$.*

THEOREM 2.4. *Let A and B be jointly tied. Then, for a given binary expansion of x, $P(A, B, x)$ exists. P is continuous at x if x is nondyadic. If A and B are jointly tied and consistent with respect to the joint eigenvalue 1, then $P(A, B, x)$ is well defined and continuous for all x.*

We establish this theorem by proving Lemmas 2.2 through 2.5.

LEMMA 2.5. *Let U be an $m \times m$ lower triangular matrix with $U_{11} = 1$, $|U_{ii}| < 1$ for $i > 1$, and $U_{ii} \ne U_{jj}$ for $i \ne j$. Then, $\lim_{\nu \to \infty} U^\nu = U^\infty$ exists, and its nonzero entries are only on the first column. Moreover, ${U^\nu}_{ij} \to 0$ exponentially for $j > 1$.*

Proof. The eigenvalues of U, $\{1, U_{22}, \ldots, U_{mm}\}$, are distinct; hence, U is diagonalizable by a similarity transformation. We write $U = S\Delta S^{-1}$, where Δ is diagonal, S and S^{-1} are lower triangular, and $\Delta_{11} = S_{11} = {S^{-1}}_{11} = 1$. Now, $U^\infty = S\Delta^\infty S^{-1}$, ${\Delta^\infty}_{11} = 1$, ${\Delta^\infty}_{ij} = 0$ for $(i,j) \neq (1,1)$. As S and S^{-1} are lower triangular, we get ${U^\infty}_{i1} = S_{i1}$, and the remaining elements of U^∞ are zero. Let $\epsilon = \max_{i>1} |U_{ii}|$. Then from $U^\nu = S\Delta^\nu S^{-1}$ we get ${U^\nu}_{ij} = O(\epsilon^\nu)$ for $j > 1$. □

Remark 2.1. The convergence ${U^\nu}_{ij} \to U^\infty$ for $j > 1$ will occur even if U_{ii} for $i > 1$ are not distinct. This is evident from the Jordan normal form of U. The convergence rate, however, could be slower. If the largest Jordan block associated with ϵ is of size q, then the elements of ${U^\nu}_{i,j}$ for $j > 1$ are at most of the order of $\binom{\nu}{q-1}\epsilon^{\nu-q+1}$.

Remark 2.2. If U is triangular and for a fixed i and any $j > i$ we have $|U_{ii}| > |U_{jj}|$, then, as $\nu \to \infty$, the ith column of $(U/U_{ii})^\nu$ converges to a finite vector and all subsequent columns tend to zero.

LEMMA 2.6. *Let A and B be jointly tied. Then, for a given binary expansion of x, $\tilde{P} = P(\tilde{A}, \tilde{B}, x)$ exists and the only nonzero entries of $\tilde{P}$ are in its first column. In particular, $\tilde{P}_{11} = 1$. $\tilde{P}$, as a function of x, is uniformly bounded.*

Proof. Let $M = \max_{i,j}\{|\tilde{A}_{ij}|, |\tilde{B}_{ij}|\}$ and $\epsilon = \max_{i>1}\{|\tilde{A}_{ii}|, |\tilde{B}_{ii}|\}$. Choose δ and $m-1$ distinct ϵ_i's such that $\epsilon < \epsilon_i < \delta < 1$ for $i > 1$. Define a lower triangular matrix U with $U_{11} = 1$, $U_{ii} = \epsilon_i$ for $i > 1$, $U_{ij} = M$ for $i > j$, and $U_{ij} = 0$ for $j > i$. The absolute values of entries of $\tilde{A}$ and $\tilde{B}$ are dominated by those of U; hence, $|P_q(\tilde{A}, \tilde{B}, x)_{ij}| \leq {U^q}_{ij}$. Now, by Lemma 2.2, U^q converges to a matrix whose nonzero elements are on its first column only. Therefore, $\lim_{q\to\infty} P_q(\tilde{A}, \tilde{B}, x)_{ij} = 0$ for $j > 1$. Moreover, as $q \to \infty$, $P_q(\tilde{A}, \tilde{B}, x)_{i1}$ appears as a series with exponentially decaying terms; hence, it converges. Specifically, denote $P_q(\tilde{A}, \tilde{B}, x)$ by $\tilde{P}_q$, and write $\tilde{P}_q - \tilde{P}_1 = \sum_{\ell=1}^{q-1} \tilde{P}_{\ell+1} - \tilde{P}_\ell = \sum_{\ell=1}^{q-1} \tilde{P}_\ell(\tilde{D} - I)$, where I is identity, $\tilde{D} = \tilde{A}$ if $x_{\ell+1} = 0$, and $\tilde{D} = \tilde{B}$ if $x_{\ell+1} = 1$. Now, $|(\tilde{D} - I)_{ij}| < M + 1$ and the exponential decay of ${U^q}_{ij}$ for $j > 1$ implies $(\tilde{P}_\ell)_{ij} = o(\delta^\ell)$ for $j > 1$. Using $(\tilde{D} - I)_{11} = 0$ we get $(\tilde{P}_\ell(\tilde{D} - I))_{ij} = o(\delta^\ell)$ for all i and j. Therefore, $\tilde{P}_q$ converges as $q \to \infty$. We have $(\tilde{P}_q)_{11} = 1$ for all q, and hence $\tilde{P}_{11} = 1$. Note that $\tilde{P}(\tilde{A}, \tilde{B}, x)$ is uniformly bounded by U^∞ for all x. □

Define Z to be the first column of $\tilde{P}$, $Z_i = P(\tilde{A}, \tilde{B}, x)_{i1}$. Note that $\tilde{P}Z = Z$ and $Z_1 = 1$. Let $W = SZ$. From $P = S\tilde{P}S^{-1}$ we get $P_{ij} = W_i {S^{-1}}_{1j}$ and $PW = W$; i.e., W is the eigenvector of P associated with eigenvalue 1. (All other eigenvalues are zero and the null space is generated by columns 2 through m of S.)

LEMMA 2.7. *Let A and B be jointly tied and consistent; then $AB^\infty = BA^\infty$.*

Proof. Since A and B are jointly tied, then, by Lemma 2.3, A^∞ and B^∞ exist. Moreover, similarity transformation preserves consistency, and $\tilde{A}$ and $\tilde{B}$ are also consistent. We have $AB^\infty - BA^\infty = S(\tilde{A}\tilde{B}^\infty - \tilde{B}\tilde{A}^\infty)S^{-1}$. Now, $\tilde{B}^\infty$ has only zeros on columns 2 through m, and the first column is just the eigenvector of $\tilde{B}$ whose first entry is 1. The same applies to $\tilde{A}$. We have $(\tilde{A}\tilde{B}^\infty - \tilde{B}\tilde{A}^\infty)_{i,j} = 0$ because, for $j > 1$, ${B^\infty}_{ij} = {A^\infty}_{ij} = 0$, and, for $j = 1$, the cancellations occur due to consistency of $\tilde{A}$ and $\tilde{B}$. □

LEMMA 2.8. *Let A and B be jointly tied and consistent; then $P(A, B, x)$ is a continuous function of x.*

Proof. We prove this first for the case when x is not dyadic and then for the case when x is dyadic. Only in the latter case do we use the consistency of A and B. The similarity transformation preserves continuity. Hence, it is sufficient to prove that $P(\tilde{A}, \tilde{B}, x)$ is a continuous function of x.

Case 1. Assume that x is not dyadic. Then, the binary expansion of x does not have a tail of zeros or a tail of ones. Hence, $y \to x$ implies that an increasing number of digits of y agree with those of x.

Suppose that y agrees with x on the first q digits; then

$$P(\tilde{A}, \tilde{B}, x) - P(\tilde{A}, \tilde{B}, y) = P_q(\tilde{A}, \tilde{B}, x)[P(\tilde{A}, \tilde{B}, \bar{x}_q) - P(\tilde{A}, \tilde{B}, \bar{y}_q)]. \tag{2.6a}$$

Now, for sufficiently large q, $P_q(\tilde{A}, \tilde{B}, x)$ has near zero entries in positions (i, j) for $j > 1$. Moreover, $[P(\tilde{A}, \tilde{B}, \bar{x}_q) - P(\tilde{A}, \tilde{B}, \bar{y}_q)]$ has a zero entry in the $(1, 1)$ position and the remaining entries are uniformly bounded. As a result, the right-hand side of (2.6a) approaches zero as $q \to \infty$. Therefore, we have $\lim_{y\to x} P(\tilde{A}, \tilde{B}, y) = P(\tilde{A}, \tilde{B}, x)$ for x nondyadic.

Case 2. Assume that x is dyadic. If y approaches x while agreeing with an increasing number of digits of x, then Case 1 applies. Otherwise let $x = 0.x_1x_2 \cdots x_q1000\cdots$ and $y = 0.x_1x_2\cdots x_q01\cdots 1y_{q+\nu+2}y_{q+\nu+3}\cdots$, where the ν digits y_{q+2} through $y_{q+\nu+1}$ are equal to 1. Note that $y \to x$ as $\nu \to \infty$, but only the first q digits of y and x agree.

Now, we write

$$P(\tilde{A}, \tilde{B}, x) - P(\tilde{A}, \tilde{B}, y) = P_q(\tilde{A}, \tilde{B}, x)[\tilde{B}\tilde{A}^\infty - \tilde{A}\tilde{B}^\nu P'], \tag{2.6b}$$

where $P' = P(\tilde{A}, \tilde{B}, \bar{y}_{q+\nu+1})$. Notice that $\lim_{\nu\to\infty} \tilde{B}^\nu P' = \tilde{B}^\infty$ since all columns of $\tilde{B}^\nu$, except the first one, approach zero while $P'_{11} = 1$ and P' stays uniformly bounded. Lemma 2.4 gives $\lim_{\nu\to\infty} \tilde{B}\tilde{A}^\infty - \tilde{A}\tilde{B}^\nu P' = \tilde{B}\tilde{A}^\infty - \tilde{A}\tilde{B}^\infty = 0$. Therefore, $\lim_{y\to x} P(\tilde{A}, \tilde{B}, y) = P(\tilde{A}, \tilde{B}, x)$ for x dyadic. ◻

This concludes the proof of Theorem 2.1.

A function f is said to have Hölder exponent (at least) α for $0 \le \alpha \le 1$ if there is $C \ge 0$ such that $|f(x) - f(y)| \le C|x - y|^\alpha$. Based on this definition, we can obtain additional regularity information about P by combining (2.6a) and (2.6b).

LEMMA 2.9. *Suppose* $1 > 2^{-r} = \delta > \max_{i>1}\{|\tilde{A}_{ii}|, |\tilde{B}_{ii}|\}$. *Then the Hölder exponent of* P *is at least* $r = -\log_2(\delta)$.

Proof. Consider $1 \ge y > x \ge 0$; then the binary expansions of x and y will be $x = 0.x_1\cdots x_q011\cdots 1x_{q+n+2}x_{q+n+3}\cdots$ and $y = 0.x_1\cdots x_q100\cdots 0y_{q+n+2}y_{q+n+3}\cdots$, where the first q digits are identical and the digits in positions $q+2$ through $q+n+1$ are ones for x and zeros for y. Moreover $x_{q+n+2} = y_{q+n+2}$ or $x_{q+n+2} = 0$ and $y_{q+n+2} = 1$. In the former case $y - x \ge 2^{-(q+n+2)}$, and in the latter case $y - x \ge 2^{-(q+n+1)}$. In either case $|y - x|^r \ge \delta^{n+q+2}$. We have $\tilde{P}(y) - \tilde{P}(x) = \tilde{P}_q(x)[\tilde{B}\tilde{A}^n\tilde{P}(\bar{y}_{q+n+1}) - \tilde{A}\tilde{B}^n\tilde{P}(\bar{x}_{q+n+1})]$. The absolute values of the (i, j) elements of $\tilde{P}(q)$ for $j > 1$ are bounded by $C_1\delta^q$ for some $C_1 > 0$. The $(1, 1)$ entry of the bracket is zero, and all others are bounded by $C_2\delta^n$. Hence $|\tilde{P}(y) - \tilde{P}(x)| \le C_1C_2\delta^{n+q} \le C|y - x|^r$, where $C = C_1C_2\delta^{-2}$. Hence the Hölder exponent of $\tilde{P}(x)$, and therefore that of $P(x)$, is at least $r = -\log_2(\delta)$. ◻

We have identified sufficient conditions for $P(A, B, x)$ to be well defined and continuous. Now we concentrate on the special matrices given by (2.2c). The following two lemmas make full use of the particular structure of T_0 and T_1. They will be used to specialize the result of Theorem 2.1 to the solution of the dilation equation (2.1).

LEMMA 2.10. *Let* G *be the matrix obtained by removing the first row and column of* T_0 *or the last row and column of* T_1. *Thus* $G_{ij} = c_{2i-j}$, $1 \le i, j \le m - 1$. *Suppose that* G *has a right eigenvector* $V = (g_1, g_2, \ldots, g_{m-1})$ *associated with eigenvalue* α*; then* T_0 *has a right eigenvector* $V_0 = (0, g_1, g_2, \ldots, g_{m-1})$ *and* T_1 *has a right* *eigenvector* $V_1 = (g_1, g_2, \ldots, g_{m-1}, 0)$*, both with the same eigenvalue* α *and satisfying* $T_0V_1 = T_1V_0$*. If* α *is simple, then* T_0 *and* T_1 *are consistent with respect to* α*. If* $\sum_k c_{2k} = \sum_k c_{2k+1} = \beta$*, then* G*,* T_0*, and* T_1 *each have a left eigenvector of the form* $(1, 1, \ldots, 1)$ *with the same eigenvalue* β*. Suppose that* β *is simple; then* T_0 *and* T_1 *are consistent with respect to the corresponding right eigenvectors and* β*.*

Proof. This is immediate from the special structure of T_0 and T_1. □

DEFINITION 2.11. *The coefficients* c_k *are said to satisfy the unit column sum rule if*

$$\sum_k c_{2k} = \sum_k c_{2k+1} = 1. \tag{2.7}$$

If c_k satisfy the unit column sum rule and 1 is a simple eigenvalue, then T_0 and T_1 will be consistent with respect to 1. If, in addition, T_0 and T_1 are jointly tied, then our construction yields a continuous solution of (2.3) and the corresponding continuous solution of (2.1). (Notice that $\Phi(0)$ and $\Phi(1)$ are eigenvectors of T_0 and T_1 corresponding to eigenvalue 1. According to Lemma 2.7, they are shift continuous, i.e., $\Phi(1)_{i-1} = \Phi(0)_i$ for $i > 1$.) Now, we proceed to show that ϕ is properly normalized.

LEMMA 2.12. *Let* ϕ *be a continuous solution of* (2.1)*, and assume* $\Gamma = \int \phi(x)dx \neq 0$*. Then*

$$\sum_k c_k = 2. \tag{2.8}$$

If $\Gamma = 1$*, then*

$$\sum_k c_{2k} \sum_n \phi(2n+1) + \sum_k c_{2k+1} \sum_n \phi(2n) = 1. \tag{2.9}$$

Moreover, if $\Gamma = 1$ *and* c_k *satisfy the unit column sum rule* (2.7)*, then for any* x

$$\sum_k \phi(k+x) = 1. \tag{2.10}$$

Proof. The first sum rule (2.8) for c_k's is obtained by integrating (2.1). (We use the compactness of the support of ϕ and c_k to simplify our formulas. Unless otherwise indicated, the integrals are over the entire reals and the summations are over the entire integers.) To establish (2.9) we form a Riemann sum for the integral and simplify the sum using (2.1).

Consider the dyadics points at a fixed level ℓ, i.e., the ones of form $(2n+1)/2^\ell$. We use these points to form a Riemann sum S_ℓ to approximate $\int \phi$. We have $S_\ell = 2^{1-\ell} \sum_n \phi((2n+1)/2^\ell)$. Now, we apply the recursion relation (2.1) to write $\phi((2n+1)/2^\ell)$ in terms of the dyadics at level $\ell - 1$. Assume $\ell > 1$; then we have

$$\begin{aligned}\sum_n \phi\left(\frac{2n+1}{2^\ell}\right) &= \sum_n \sum_k c_k \phi\left(\frac{2n+1}{2^{\ell-1}} - k\right) \\ &= \sum_k c_k \sum_n \phi\left(\frac{2n+1-k2^{\ell-1}}{2^{\ell-1}}\right) = \sum_k c_k \sum_n \phi\left(\frac{2n+1}{2^{\ell-1}}\right).\end{aligned}$$

Therefore, $S_\ell = 1/2\sum_k c_k S_{\ell-1}$. Hence, if $\ell > 1$ and $\sum c_k = 2$, then $S_\ell = S_{\ell-1}$. However, if $\ell = 1$, we get

$$\begin{aligned} S_1 &= \sum_n \phi\left(\frac{2n+1}{2}\right) = \sum_n \sum_k c_k \phi(2n+1-k) \\ &= \sum_k c_{2k} \sum_n \phi(2n+1-2k) + \sum_k c_{2k+1} \sum_n \phi(2n+1-(2k+1)) \\ &= \sum_k c_{2k} \sum_n \phi(2n+1) + \sum_k c_{2k+1} \sum_n \phi(2n). \end{aligned}$$

Hence, $S_\ell = S_1 = \sum_k c_{2k} \sum_n \phi(2n+1) + \sum_k c_{2k+1} \sum_n \phi(2n)$. Now, as $\ell \to \infty$, we have $S_\ell \to \int \phi = 1$, which proves (2.9). Moreover, if c_k satisfy the unit column sum rule, then we get $\sum_\ell \phi(\ell) = 1$. (One uses this result to normalize the eigenvectors of T_0 and T_1 corresponding to eigenvalue 1 in (2.4a). That is, $\sum_j \Phi(0)_j = \sum_j \Phi(1)_j = 1$.)

Finally, we prove (2.10) and show that the integral of ϕ equals the sum of ϕ at any translate of the integers. Consider a vector $V = (v_1, v_2, \ldots, v_m)^t$. From (2.7), one can easily see $\sum_i (T_0 V)_i = \sum_i (T_1 V)_i = \sum_i V_i$. Hence, if we start with $V = \Phi(0)$ or $V = \Phi(1)$ and multiply on the left with T_0's or T_1's, then at any stage the resulting values of $\phi(x)$ at dyadics satisfy (2.10), and in the limit the same equation is satisfied at all points by continuity of ϕ. □

THEOREM 2.13. *If T_0 and T_1 are jointly tied and their entries c_k satisfy the unit column sum rule, then $P(T_0, T_1, x)$ is well defined and continuous, the columns of P are identical, and a solution of* (2.1) *is given by $\phi(x+i-1) = P(T_0, T_1, x)_{i,j}$ for any j. Moreover, this ϕ is properly normalized, i.e., $\int \phi dx = 1$.*

Proof. Since c_k's satisfy the unit column sum rule and 1 is a simple eigenvalue, then, by Lemma 2.7, T_0 and T_1 are consistent. The matrices are assumed to be jointly tied; therefore, by Lemma 2.5, P is well defined and continuous. Since c_k's satisfy the unit column sum rule, then, by the argument in the proof of (2.10), the sum of elements of any column of any product of T_0's and T_1's, e.g., $P(T_0, T_1, x)$, is 1. Now, by the comments following Lemma 2.3, we have $P_{ij} = W_i {S^{-1}}_{1j}$ and $1 = \sum_i P_{ij} = {S^{-1}}_{1j} \sum W_i$ for any j. Hence, the elements of the first row of S^{-1} are equal, and we may assume ${S^{-1}}_{1j} = 1$. Then, the columns of P and W are equal, and each represents ϕ through $\phi(x+i-1) = P(T_0, T_1, x)_{i,j}$ for any j. Using Lemma 2.8 we get $\int \phi = \sum_i \phi(x+i-1) = \sum_i P_{ij} = 1$. Hence, ϕ is properly normalized. □

2.3. Infinite products of a finite family of matrices. Theorem 2.1 can be generalized to include the products of a finite family of matrices. Let $R > 1$ be an integer and r be a digit in base R, i.e., $0 \le r \le R-1$. Consider R matrices A_0, $A_1, \ldots, A_{R-1}$. Represent $x \in [0,1]$ by its expansion in base R, $x = 0.x_1x_2\cdots$ (now $0 \le x_q \le R-1$). Define $P_q(x) = \prod_{\ell=1}^q A_{x_\ell}$ and $P(x) = \lim_{q\to\infty} P_q(x)$ whenever the limit exists. The family is called consistent if there are simple eigenvectors V_0 and V_{R-1} such that $A_0V_0 = V_0$, $A_{R-1}V_{R-1} = V_{R-1}$, and $A_rV_{R-1} = A_{r+1}V_0$ for $0 \le r \le R-2$. Now, we have the following theorem.

THEOREM 2.14. *Let the family $\{A_r\}$ be jointly tied and consistent. Then, $P(x)$ exists and is continuous.*

Proof. This is similar to Theorem 2.1. □

2.4. Analysis of three-term dilation equations. In this section we give an example based on the case $m = 2$. We can achieve triangularization if $c_1 = c_0 + c_2$, in which case for any $a \neq b$ we have

$$T_0 = \begin{pmatrix} c_0 & 0 \\ c_2 & c_1 \end{pmatrix} = \frac{1}{a-b}\begin{pmatrix} a & -1 \\ -b & 1 \end{pmatrix}\begin{pmatrix} c_1 & 0 \\ ac_2 & c_0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ b & a \end{pmatrix},$$

$$T_1 = \begin{pmatrix} c_1 & c_0 \\ 0 & c_2 \end{pmatrix} = \frac{1}{a-b}\begin{pmatrix} a & -1 \\ -b & 1 \end{pmatrix}\begin{pmatrix} c_1 & 0 \\ bc_0 & c_2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ b & a \end{pmatrix}.$$

We note that the triangularization is not unique. We set $a = 0$ and $b = 1$ to get

$$T_0 = \begin{pmatrix} c_0 & 0 \\ c_2 & c_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c_1 & 0 \\ 0 & c_0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$T_1 = \begin{pmatrix} c_1 & c_0 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c_1 & 0 \\ c_0 & c_2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now, $\tilde{T}_0$ and $\tilde{T}_1$ are the middle matrices on the right-hand side of previous equations. Given an x it is easy to form $\tilde{P} = P(\tilde{T}_0, \tilde{T}_1, x)$. For $c_1 = 1$, $0 < c_0 < 1$, and $0 < c_2 < 1$, we have $\tilde{P}_{11} = 1$, $\tilde{P}_{12} = \tilde{P}_{22} = 0$. Define $\sigma_q = \sigma_q(x) = \sum_{n=1}^{q} x_n$ and $\sigma_0(x) = 0$. Then, $(\tilde{P}_{q+1})_{21} = (\tilde{P}_q)_{21} + x_{q+1}c_0(\tilde{P}_q)_{22}$ and $(\tilde{P}_q)_{22} = {c_0}^{q-\sigma_q}{c_2}^{\sigma_q}$. Therefore,

$$\tilde{P}_{2,1} = z(x) = \sum_{q=0}^{\infty} x_{q+1}{c_0}^{q+1-\sigma_q}{c_2}^{\sigma_q}.$$

Now, using our previous notation (following Lemma 2.3) we have $Z(x) = (1, z(x))^t$ and

$$P(T_0, T_1, x) = \begin{pmatrix} z(x) & z(x) \\ 1-z(x) & 1-z(x) \end{pmatrix}, \quad \phi(x) = \begin{cases} z(x) & \text{for } 0 \le x \le 1, \\ 1 - z(x-1) & \text{for } 1 \le x \le 2. \end{cases}$$

It is easy to verify that $\phi(x)$ is increasing on $[0, 1]$ and decreasing on $[1, 2]$.

2.5. Simultaneous triangularization. The main step in our analysis of the products of two matrices is to reduce them to a triangular form. Given A and B, we search for $\tilde{A}$, $\tilde{B}$, and S^{-1} such that $S^{-1}A = \tilde{A}S^{-1}$ and $S^{-1}B = \tilde{B}S^{-1}$. This constitutes $2m^2$ nonlinear algebraic equations. In the case of wavelets, one always enforces (2.7). Then, ${S^{-1}}_{1j} = \tilde{A}_{11} = \tilde{B}_{11} = 1$. This reduces the number of equations to $2m^2 - 2m$ and the number of unknowns to $2m^2 - 2$. (The eigenvalues of A and B are the elements on the diagonal of $\tilde{A}$ and $\tilde{B}$, but their positions are not known.) Therefore, there are $2m - 2$ degrees of freedom in the triangularization (e.g., a and b in section 2.4). Despite the presence of degrees of freedom, simultaneous triangularization is rarely possible. (It is known that a family of matrices is simultaneously triangularizable iff the eigenvalues of any product of matrices are equal, in some order, to the products of eigenvalues of the same matrices.)

The triangularizer matrices which are useful for the construction of N-dimensional scaling functions are the ones which work for a class of matrices and have constant entries. For example, if $m = 3$, then T_0 and T_1 can be triangularized when $c_0 + c_3 = 1$ or $1/2$. But if the sum is 1, then the triangularizer depends on c_0 and will not be suitable for higher-dimensional constructions considered in this paper. On the other hand when the sum is $1/2$, then the triangularizer is constant. In the next section we focus on the latter case.

2.6. Analysis of $(m+1)$-term dilation equations. A class of matrices T_0 and T_1 for which constant triangularizers have been obtained are exactly those which satisfy certain sum rules used to enforce high regularity [9]. Here we require a particular subset of such rules, i.e.,

$$\sum_k c_k k^q (-1)^k = 0 \quad \text{for} \quad q = 0, \ldots, m-2, \tag{2.11}$$

where 0^0 is taken to be 1. One can solve (2.11) for $c_1, \ldots, c_{m-1}$ in terms of the "corner" values c_0 and c_m. The resulting coefficients c_k and the associated matrices T_0 and T_1 satisfy a host of binomial-type identities. The outlines for the proof of some of these identities are collected in Appendix A as notes. In what follows, we adopt the usual conventions that $\binom{a}{b} = 0$ if $b > a$ or $b < 0$, and $1/c! = 0$ if $c < 0$.

The coefficient matrix of (2.11) is a Vandermonde-type matrix with a nonzero determinant. Therefore, it is nonsingular. The unique solution is given by "binomial interpolation" between the endpoint values (see Note A.1),

$$c_k = c_0 \binom{m-1}{k} + c_m \binom{m-1}{k-1}. \tag{2.12}$$

For this particular choice of c_k's, an $m \times m$ triangularizer matrix S and its inverse S^{-1} are given by (see Note A.2.)

$$S_{ij} = \binom{j-1}{m-i} \frac{(-1)^{i+j-m-1}}{(j-1)!}, \qquad S^{-1}_{ij} = \binom{m-j}{i-1} (i-1)!. \tag{2.13}$$

(Notice that the entries of S and S^{-1} are zero below the second diagonal, i.e., if $i+j > m+1$.) Upon triangularization, the diagonal elements of $\tilde{T_0}$ and $\tilde{T_1}$, respectively, are (see Note A.2)

$$\begin{array}{l} 2^{m-2}(c_0 + c_m), 2^{m-3}(c_0 + c_m), \ldots, (c_0 + c_m), c_0, \\ 2^{m-2}(c_0 + c_m), 2^{m-3}(c_0 + c_m), \ldots, (c_0 + c_m), c_m. \end{array} \tag{2.14}$$

By Theorem 2.1, we will have a continuous scaling function if the leading eigenvalue is one and the remaining eigenvalues are less than one in absolute value. Therefore, we will have a continuous scaling function if

$$c_0 + c_m = 1/2^{m-2}, \quad |c_0| < 1, \quad |c_m| < 1, \tag{2.15}$$

and the remaining c_k's are determined by (2.12). We summarize this result in the following theorem.

THEOREM 2.15. *If*

$$c_0 + c_m = 1/2^{m-2}, \quad |c_0| < 1, \quad |c_m| < 1,$$

and

$$c_k = c_0 \binom{m-1}{k} + c_m \binom{m-1}{k-1},$$

then $P(T_0, T_1, x)$ *gives the normalized continuous solution of* (2.1).

2.7. Analysis of smooth scaling functions. If the inequalities in (2.15) are made stricter by a factor of $1/2^\ell$, then the degree of smoothness of ϕ increases by ℓ . This is expressed in the following theorem.

THEOREM 2.16. *If*

$$c_0 + c_m = 1/2^{m-2}, \quad |c_0| < 1/2^\ell, \quad |c_m| < 1/2^\ell$$

for an integer $0 \le \ell < m-1$ *and*

$$c_k = c_0 \binom{m-1}{k} + c_m \binom{m-1}{k-1},$$

then $P(T_0, T_1, x)$ *gives the normalized* ℓ *times continuously differentiable solution of* (2.1).

Proof. This can be shown by considering the divided difference of ϕ:

$$\Delta(\phi, \ell, h, x) = \frac{1}{h^\ell} \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \phi(x + (\ell - i)h). \tag{2.16}$$

If the limit of above expression, as $h \to 0$, exists, then ϕ is ℓ times differentiable. We will use the matrix form of (2.16) and some of the results from Theorem A.1 (see Appendix A) to prove this theorem.

Consider the binary expansion of the numbers $x + jh$ for $0 \le j \le \ell$. If x is not a dyadic, then, as $h \to 0$, the number of common initial digits of the numbers $x + jh$ will tend to infinity. To ensure the same for dyadic x, we use the expansion of x that ends in a tail of zeros if $h > 0$; however, if $h < 0$, then we use the expansion that ends in a tail of ones. Suppose that the binary expansions of $x + jh$'s differ only on the k (possibly infinite) digits in the positions $n+1$ through $n+k$. Then we may write $x + jh = y + 2^{-n} w_j + 2^{-n-k} z$, where w_j's, for $0 \le j \le \ell$, are equidistant numbers, y represents the initial common digits, and z represents the ending common digits (if any). Here y, z, and w_j's are in the unit interval. Let $\hbar = 2^n h$, $\theta = w_0$, and define $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}(k, \ell, \hbar, \theta)$, as in Theorem A.1, by

$$\tilde{\mathcal{D}} = \tilde{\mathcal{D}}(k, \ell, \hbar, \theta) = \frac{1}{\hbar^\ell} \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \tilde{P}_k(w_{\ell-i}). \tag{2.17}$$

Then the triangularization of the matrix form of (2.16) leads to

$$\tilde{Q}_n(x) = S^{-1} \Delta(\Phi, \ell, h, x) S = 2^{n\ell} \tilde{P}_n(x) \tilde{\mathcal{D}}(k, \ell, \hbar, \theta) \tilde{P}(z). \tag{2.18}$$

While the first ℓ columns of $2^{n\ell}\tilde{P}_n(x)$ grow unbounded as $n \to \infty$, they are nullified by the first ℓ rows of $\tilde{\mathcal{D}}$, which are zero. The $(\ell+1)$-st column of $2^{n\ell}\tilde{P}_n(x)$ is finite, and its diagonal entry is 1. This column multiplies the first entry of the $(\ell+1)$-st row of $\tilde{\mathcal{D}}$, which is $\ell!$. The remaining columns of $2^{n\ell}\tilde{P}_n(x)$ for $j > \ell + 1$ and elements of $\tilde{\mathcal{D}}$ for $i - j < \ell$ are zero. Also notice that $\tilde{P}(z)_{11} = 1$ and $\tilde{P}(z)_{ij} = 0$ for $j > 1$. As a result, all columns of $\tilde{Q} = \lim_{n\to\infty} \tilde{Q}_n$ beyond the first one are zero. Moreover, the first column is simply the "normalized" form of the $(\ell+1)$-st column of $\tilde{P}(x)$, that is,

$$\tilde{Q}(x)_{i,1} = \ell! \lim_{n\to\infty} 2^{n\ell} \tilde{P}_n(x)_{i,\ell+1}. \tag{2.19}$$

The existence of this limit follows from Remark 2.2.

To establish the continuity of the ℓth derivative, we note that the pair of eigenvectors of $\tilde{T}_0$ and $\tilde{T}_1$ corresponding to the eigenvalue $1/2^\ell$ are, by Lemma 2.7, consistent and shift continuous. Then an argument similar to Lemma 2.5 or Theorem 2.2 shows that $\tilde{Q}$ is continuous and shift continuous. Hence ϕ is ℓ times continuously differentiable. ◻

Remark 2.3. If ℓ is allowed to be a real number, $\ell = [\ell] + r$, $0 \le r < 1$, then $d\phi^{[\ell]}/dx^{[\ell]}$ is Hölder continuous with exponent at least r. This follows from considering the submatrices obtained by removing the first $[\ell]$ rows and columns of $2^{[\ell]}\tilde{T}_0$, $2^{[\ell]}\tilde{T}_1$, and applying the methods of Remark 2.2 and Lemma 2.6.

3. *N*-dimensional scaling functions. Higher-dimensional wavelets and scaling functions are important in analyzing multivariable cases. Rectangular wavelets can be constructed for $\mathcal{R}^N$ in ways similar to the one-dimensional case [7, Chapter 10]. However, as the number of dimensions and coefficients increases, it becomes less practical to ascertain regularity properties of the general N-dimensional scaling functions and the corresponding wavelets.

Our aim in this section is to identify a class of smooth scaling functions by generalizing the results from section 2 to N dimensions. In order to abbreviate the formulas and compare quantities in N dimensions, we first introduce a few notations. Assume $X = (X_1, X_2, \ldots, X_N)$ and $Y = (Y_1, Y_2, \ldots, Y_N)$ are two N-tuples. We define *reverse lexicographic order* $X \prec Y$ to mean $X_N < Y_N$, or there is $1 \le n < N$ so that $X_n < Y_n$ and $X_{n'} = Y_{n'}$, for $n' > n$. For the N-tuple $I \in \{1, \ldots, m\}^N$, we define $\hat{I} = 1 + \sum_{n=1}^{N} m^{n-1}(I_n - 1)$. The hat function enumerates the cells in the N-cube, $[0, m]^N$, from 1 to m^N, by going through components with lower indices first. Notice $\hat{I} < \hat{J}$ iff $I \prec J$. Finally, if s is a scalar, and it is added or compared to a vector, then s stands for $(s, s, \ldots, s)$.

Now, to generalize (2.2) to N dimensions, let $H_\mu \in \{0, \ldots, m-1\}^N$ for $\mu = 1, \ldots, m^N$ be ordered by $(0, \ldots, 0) = H_1 \prec H_2 \prec \cdots \prec H_{m^N} = (m-1, \ldots, m-1)$. Then we have

$$\Phi(X) = [\phi(X + H_1), \phi(X + H_2), \ldots, \phi(X + H_{m^N})]^t \quad \text{for} \quad X \in [0,1]^N, \tag{3.1a}$$

$$C_K = 0 \quad \text{for} \quad K \notin \{0, \ldots, m\}^N, \tag{3.1b}$$

$$(T_D)_{\hat{I}\hat{J}} = C_{2I-J+D-1} \quad \text{for} \quad I, J \in \{1, \ldots, m\}^N \quad \text{and} \quad D \in \{0,1\}^N. \tag{3.1c}$$

(In what follows, the range of I, J, and D is as in (3.1c) unless further restricted.) Notice that T_D is an $m^N \times m^N$ matrix. To identify a particular entry for a given I and J, first we divide the matrix into $m \times m$ subsquares and locate the square at the position (I_N, J_N); then we divide this $m^{N-1} \times m^{N-1}$ matrix into m by m subsquares and locate the square at the position (I_{N-1}, J_{N-1}) and so on until (I_1, J_1) is located in the final $m \times m$ matrix. In this manner we have N nested grids on each T_D. We label these grids as *level* N (for the coarsest) through level 1 (for the finest). The value of D_n determines the nth component of the index of C_K. Inside each level-n grid element this component is fixed, and across the grid elements its values changes in a pattern similar to the indexing of the matrix for the one-dimensional problem, i.e., T_{D_n}. The triangularization steps (see section 3.2 below) will utilize these grids. The statements and proofs of the one-dimensional case are easily generalized to the N-dimensional case by using these grids.

To generalize the iteration formula (2.3), first we define the following convention. We will show the qth digit of the nth component of X by $X_{n,q}$. Then, $X_{*,q}$ will indicate the vector of such digits. Similarly, $\bar{X}_{n,q}$ and $\bar{X}_{*,q}$ will be used to indicate the residual after the qth digit. Now, any continuous solution of (1.1) with its support in $[0,m]^N$ satisfies

$$\Phi(X) = T_{X_{*,1}}\Phi(2X - X_{*,1}) = T_{X_{*,1}}\Phi(\bar{X}_{*,1}), \tag{3.2}$$

and the repeated applications of (3.2) result in

$$\Phi(X) = \prod_{\ell=1}^{q} T_{X_{*,\ell}}\Phi(\bar{X}_{*,q}). \tag{3.3}$$

We define $P_q(\{T_D\}, X) = \prod_{\ell=1}^{q} T_{X_{*,\ell}}$, and $P(\{T_D\}, X) = \lim_{q\to\infty} P_q(\{T_D\}, X) =$ whenever the limit exits.

Suppose $X \in [0,1]^N$ and $1 \le n \le N$, and define $X\rceil n$ (respectively, $X\rfloor n$) to be a vector which is same as X except that its nth component is 1 (respectively, 0). We define $F(X) : [0,1]^N \to \mathcal{R}^{m^N}$ to be shift continuous if for any n, $1 \le n \le N$, $F(X\rfloor n)_{\hat{I}} = F(X\rceil n)_{\hat{J}}$ whenever $I - J = 0\rceil n$. (Here, the cell I is immediately after the cell J in the direction of the nth axis. The corresponding components of F are required to have the same value on the common face between the two cells.) Now, $\phi(X)$ is continuous on $[0,m]^N$ iff $\Phi(X)$ is continuous on $[0,1]^N$ and shift continuous. (As Φ is determined from its values at $\{0,1\}^N$, the requirement of shift continuity may also be limited to this set.)

The notion of consistency of matrices (as it appears in Definition 2.2 for base 2 and in Theorem 2.3 for bases larger than 2) can be generalized to higher dimensions in a componentwise fashion. For example, consider a set of matrices A_D for $D \in \{0,1\}^N$. These matrices are called consistent if for any n, $1 \le n \le N$, the pair $A_{D\rfloor n}, A_{D\rceil n}$ are consistent.

THEOREM 3.1. *Suppose that $\{A_D\}$ are jointly tied; then, for a given binary expansion of X, $P(\{A_D\}, X)$ exists. P is well defined and continuous at X if all components of X are nondyadic. If the matrices are consistent, then P is well defined and continuous at any X. P is Hölder continuous with exponent at least $-\log_2(\delta)$ if $1 > \delta > |\lambda|$, where λ is any nonleading eigenvalue of any A_D.*

Proof. The existence of P is proved in the same manner as in the one-dimensional case (Theorem 2.1). To prove continuity or obtain the Hölder exponent, we estimate $|P(Y) - P(X)|$ through the triangle inequality. Consider an N-cube with X and Y as two diagonally opposite corners. Define a set of points Z_n, $1 \le n \le p = 2^{N-2} + 2$, which start at X, go through the vertices of the N-cube, and arrive at Y. We have $|P(Y) - P(X)| \le \sum_{n=1}^{p-1} |P(Z_{n+1}) - P(Z_n)|$. Each consecutive pair of vertices differ in only one coordinate, and hence, we may apply the estimates in Lemma 2.5 or 2.6 to each term of the sum. (The same estimates cannot be applied to $P(Y) - P(X)$ directly because different components of X and Y may approach each other at different rates or some components may be dyadic while others are nondyadic.) The remaining steps in the proof are similar to the one-dimensional case. □

Suppose that Ω is a sublist of $(1, \ldots, N)$, i.e., $\Omega = (n_1, n_2, \ldots)$ and $1 \le n_1 < n_2 < \cdots \le N$. We say K is even (odd) on Ω if $K_{n_1}, K_{n_2}, \ldots$ are even (odd) and the remaining components of K are odd (even). Now the sum rule (2.7) can be generalized as follows.

DEFINITION 3.2. *We say the coefficients C_K satisfy the unit column sum rule if*

$$\sum_{K \text{ odd on } \Omega} C_K = \sum_{K \text{ even on } \Omega} C_K = 1 \quad \textit{for every } \Omega. \tag{3.4}$$

Notice that if C_K's satisfy this property, then the column sum for any column of any T_D is 1. In that case, all matrices have a left eigenvector of the form $(1, 1, \ldots, 1)$. As in the one-dimensional case, there is a matrix G, obtained by eliminating certain rows and columns of the matrices, which has a similar left eigenvector. To obtain G, we start from any T_D and for each n eliminate the first (respectively, the last) row and column of each of the level-n grid elements if D_n is zero (respectively, one). Thus G is an $(m-1)^N \times (m-1)^N$ matrix and is given by $G_{\hat{I}', \hat{J}'} = C_{2I'-J'}$, where I' and J' are taken from $\{1, \ldots, m-1\}^N$ and $\hat{I}' = 1 + \sum_{n=1}^{N} (m-1)^{n-1}(I'_n - 1)$. Any right eigenvector of G can be extended to an eigenvector of T_D by padding it with zeros at the locations where the rows of T_D were eliminated. In particular the right eigenvector corresponding to 1 generates an eigenvector for each T_D. If 1 is a simple eigenvalue, then the consistency of $\{T_D\}$ easily follows.

We are looking for the unique normalized continuous solution of (1.1) with support in $[0, m]^N$. According to (3.2) such a solution will satisfy certain simple and important properties (similar to (2.4)) as follows:

$$T_D \Phi(D) = \Phi(D), \tag{3.5a}$$

$$\Phi\left(\frac{D\rfloor n + D\rceil n}{2}\right) = T_{D\rfloor n}\Phi(D\rceil n) = T_{D\rceil n}\Phi(D\rfloor n), \tag{3.5b}$$

and in general for any $X \in [0, 1]^N$

$$\Phi\left(\frac{X\rfloor n + X\rceil n}{2}\right) = T_{X_{*,1}\rceil n}\Phi(\bar{X}_{*,1}\rfloor n) = T_{X_{*,1}\rfloor n}\Phi(\bar{X}_{*,1}\rceil n). \tag{3.5c}$$

Other properties of these solutions (similar to the "sum rules" in Lemma 2.8) are expressed as follows.

LEMMA 3.3. *Let ϕ be a continuous solution of* (1.1), *and assume* $\Gamma = \int \phi(X)dX \neq 0$. *Then*

$$\sum_K C_K = 2^N. \tag{3.6}$$

If $\Gamma = 1$, *then*

$$\sum_{\Omega} \left(\sum_{K \text{ even on } \Omega} C_K \sum_{K \text{ odd on } \Omega} \phi(K) + \sum_{K \text{ odd on } \Omega} C_K \sum_{K \text{ even on } \Omega} \phi(K) \right) = 1. \tag{3.7}$$

Moreover, if C_K's satisfy the unit column sum rule and $\Gamma = 1$, *then for all X we have*

$$\sum_K \phi(X + K) = 1. \tag{3.8}$$

Proof. This is identical to the proof for the one-dimensional case. The integration of (1.1) gives (3.6). A Riemann sum approximation to the integral provides (3.7) and its special case (3.8). ∎

THEOREM 3.4. *If C_K's satisfy the unit column sum rule and T_D's are jointly tied, then $P(\{T_D\},X)$ exists, is continuous, and has identical columns. The dilation equation* (1.1) *has a continuous solution given by $\phi(X+I-1)=P(\{T_D\},X)_{\hat{I},\hat{J}}$ for any $\hat{J}$. Moreover, this solution is properly normalized, i.e., $\int \phi(X)dX = 1$.*

Proof. The proof is identical to the proof for the one-dimensional case, Theorem 2.2. ◻

3.1. Analysis of 3^2-term dilation equations. In this section, we give an example based on $m = 2$ and $N = 2$. We have $\Phi(X_1,X_2) = [\phi(X_1,X_2), \phi(X_1+1,X_2), \phi(X_1,X_2+1), \phi(X_1+1,X_2+1)]^t$ and

$$T_{00} = \begin{pmatrix} C_{00} & 0 & 0 & 0 \\ C_{20} & C_{10} & 0 & 0 \\ C_{02} & 0 & C_{01} & 0 \\ C_{22} & C_{12} & C_{21} & C_{11} \end{pmatrix}, \quad T_{01} = \begin{pmatrix} C_{01} & 0 & C_{00} & 0 \\ C_{21} & C_{11} & C_{20} & C_{10} \\ 0 & 0 & C_{02} & 0 \\ 0 & 0 & C_{22} & C_{12} \end{pmatrix},$$

$$T_{10} = \begin{pmatrix} C_{10} & C_{00} & 0 & 0 \\ 0 & C_{20} & 0 & 0 \\ C_{12} & C_{02} & C_{11} & C_{01} \\ 0 & C_{22} & 0 & C_{21} \end{pmatrix}, \quad T_{11} = \begin{pmatrix} C_{11} & C_{01} & C_{10} & C_{00} \\ 0 & C_{21} & 0 & C_{20} \\ 0 & 0 & C_{12} & C_{02} \\ 0 & 0 & 0 & C_{22} \end{pmatrix}.$$

We can triangularize these matrices in two steps if

$$\begin{aligned} &C_{10} = C_{00} + C_{20},\ C_{01} = C_{00} + C_{02},\ C_{12} = C_{02} + C_{22},\ C_{21} = C_{20} + C_{22}, \\ &C_{11} = C_{01} + C_{21} = C_{10} + C_{12} = C_{00} + C_{20} + C_{22} + C_{02}. \end{aligned} \tag{3.9}$$

Let $\mathbf{I}_n$ denote an $n \times n$ identity matrix, and define

$$S = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & \mathbf{I}_2 \\ \mathbf{I}_2 & -\mathbf{I}_2 \end{pmatrix}, \quad S_1 = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix},$$

$$S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad {S_2}^{-1} = \begin{pmatrix} \mathbf{I}_2 & \mathbf{I}_2 \\ \mathbf{I}_2 & 0 \end{pmatrix}, \quad {S_1}^{-1} = \begin{pmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix}.$$

Then, any of matrices T_D, $D \in \{0,1\}^2$, can be triangularized by

$$\tilde{T}_D = {S_1}^{-1} {S_2}^{-1} T_D S_2 S_1.$$

The four eigenvalues of each of the four matrices may be given in terms of the "corner" C_K's as

$$\begin{aligned} &(C_{00} + C_{20} + C_{22} + C_{02},\ C_{00} + C_{02},\ C_{00} + C_{20},\ C_{00}), \\ &(C_{00} + C_{20} + C_{22} + C_{02},\ C_{00} + C_{02},\ C_{02} + C_{22},\ C_{02}), \\ &(C_{00} + C_{20} + C_{22} + C_{02},\ C_{22} + C_{20},\ C_{00} + C_{20},\ C_{20}), \\ &(C_{00} + C_{20} + C_{22} + C_{02},\ C_{22} + C_{20},\ C_{02} + C_{22},\ C_{22}). \end{aligned}$$

Notice that sums of C_K's on every corner, corners of every side, and corners of the entire square show up as eigenvalues. For convergence of the products of T_D's, we need the leading eigenvalue to be 1 and the remaining eigenvalues to be less than 1 in absolute value. The result can be displayed in terms of a three-parameter space, say, (C_{00}, C_{02}, C_{20}). Therefore, we require $C_{22} = 1 - C_{00} - C_{02} - C_{20}$ and

$$-1 < C_{00} < 1,\ -1 < C_{02} < 1,\ -1 < C_{20} < 1,$$
$$0 < C_{00} + C_{02} < 1,\ 0 < C_{00} + C_{20} < 1,\ 0 < C_{00} + C_{02} + C_{20} < 2.$$

The remaining C_K's are then determined by (3.9). Notice that this solution has three degrees of freedom and can change its sign on $[0, 2]^2$, while a tensor product solution $\phi(x, y) = \phi_1(x)\phi_2(y)$, where ϕ_1 and ϕ_2 satisfy

$$\phi_1(x) = \alpha\phi_1(2x) + \phi_1(2x-1) + (1-\alpha)\phi_1(2x-2),$$
$$\phi_2(y) = \beta\phi_2(2y) + \phi_2(2y-1) + (1-\beta)\phi_2(2y-2),$$

has only two degrees of freedom, i.e., $0 < \alpha, \beta < 1$, and a fixed sign.

3.2. Analysis of $(m+1)^N$-term dilation equations. In this section, we consider the dilation equations for a given m and N, with coefficients that satisfy (2.11) along every coordinate direction. In this case the coefficients, C_K, are given by binomial interpolation of their values on the corners of the N-cube, that is, C_{mD}'s. Applying (2.12) repeatedly along all coordinate directions gives

$$C_K = \sum_{D \in \{0,1\}^N} C_{mD} \prod_{n=1}^{N} \binom{m-1}{K_n - D_n}. \tag{3.10}$$

Now, all 2^N corresponding T_D's can be triangularized by a set of matrices built from S given by (2.13). These matrices are constructed as follows: given an n, $1 \le n \le N$, first replace every entry S_{ij} of S with a diagonal matrix $S_{ij}\mathbf{I}_{m^n}$. This will produce an $m^{n+1} \times m^{n+1}$ intermediate matrix. Then, use m^{N-n-1} copies of the intermediate matrix to create S_n, a block diagonal matrix of size $m^N \times m^N$. Now, we have simultaneous triangularization by

$$\tilde{T}_D = {S_1}^{-1}{S_2}^{-1} \cdots {S_{N-1}}^{-1}{S_N}^{-1} T_D S_N S_{N-1} \cdots S_2 S_1.$$

Here, at each stage n, $n = N, \ldots, 1$, the effect of S_n and ${S_n}^{-1}$ is to triangularize the current level-n grid elements. The formation of the resulting diagonal blocks is similar to the one-dimensional formula (2.14). The entries that act as c_0 and c_m are a pair of blocks, within each grid's $m \times m$ subdivision, in the positions $(1, 1 + D_n)$ and $(m, m + D_n - 1)$. We call these the *polar* blocks. At the end of each stage of triangularization the polar blocks and their sum, scaled by factors $1, \ldots, 2^{m-2}$, appear on the diagonal. The first entry of T_D is C_D, and the corresponding indices of C_K's in each pair of polar blocks differ in only one component. As a result, the final summation of C_K's occur on the corners of the faces of the N-cube. If the face is n-dimensional, then the sum will appear with scale factors of up to $2^{(m-2)n}$.

To formally describe the eigenvalues of T_D's, i.e., the diagonal elements of $\tilde{T}_D$'s, we need to construct sums of C_K's on the corners of every n-dimensional face of the N-cube, $\{0, m\}^N$. Let θ be a sublist of $(1, \ldots, N)$, $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, $1 \le \theta_1 < \theta_2 < \cdots < \theta_n \le N$, and assume that θ' is its complimentary list, $\theta \cup \theta' = \{1, \ldots, N\}$. (If θ is empty, then $n = 0$.) Now, let $D_\theta = (D_{\theta_1}, D_{\theta_2}, \ldots, D_{\theta_n})$ be a free element of $\{0, 1\}^n$, and assume that the remaining elements of D form a fixed element of $\{0, 1\}^{N-n}$, say, $D_{\theta'} = \Upsilon$. Then, for every Υ there are diagonal entries (with various multiplicities) of

$\tilde{T}_D$ in the form

$$2^\gamma \sum_{\substack{D_\theta \\ D_{\theta'}=\Upsilon}} C_{mD} \quad \text{for} \quad 0 \le \gamma \le (m-2)n.$$

Hence, for the convergence of the matrix products to a continuous function, we require

$$\sum_D C_{mD} = 1/2^{(m-2)N}, \tag{3.11}$$

which restricts the sum of C_K's on all corners of the N-cube. Similarly, for every n-dimensional ($n < N$) face of the cube, described by an Υ, we require

$$\left| \sum_{\substack{D_\theta \\ D_{\theta'}=\Upsilon}} C_{mD} \right| < 1/2^{(m-2)n}. \tag{3.12}$$

Now, (3.10)–(3.12) characterize a class of C_K's which produce continuous scaling functions. As (3.11) indicates, there are $2^N - 1$ degrees of freedom. (For each $n < N$ there are $\binom{N}{n} 2^{N-n}$ inequalities of the form (3.12). There are $3^N - 1$ inequalities in all.)

As in the case of the one-dimensional scaling functions, one can increase the degree of smoothness of ϕ by ℓ if the inequalities (3.12) are made stricter by a factor of $1/2^\ell$. We summarize our results in the following theorem.

THEOREM 3.5. *If the sum of C_K's on the corners of the N-cube satisfies*

$$\sum_D C_{mD} = 1/2^{(m-2)N},$$

and for every n-dimensional ($n < N$) face of the N-cube we have

$$\left| \sum_{\substack{D_\theta \\ D_{\theta'}=\Upsilon}} C_{mD} \right| < 1/2^{(m-2)n+\ell},$$

and all other C_K's are given by binomial interpolation of their values at the corners of the N-cube,

$$C_K = \sum_{D \in \{0,1\}^N} C_{mD} \prod_{n=1}^{N} \binom{m-1}{K_n - D_n},$$

then the solution of (1.1) *is ℓ times continuously differentiable. If ℓ is allowed to be a real number, then the $[\ell]$th derivative of ϕ is Hölder continuous with exponent at least $\ell - [\ell]$.*

Proof. If $\ell = 0$, then we establish continuity by using Theorem 3.2. For $\ell > 0$ we investigate existence, continuity, and the Hölder exponent of the required derivative of ϕ by considering the related partial derivatives. The treatment is analogous to the one-dimensional case, and it uses the generalization of Theorem A.1 to N dimensions. ∎

3.3. Examples of 4^2-term scaling functions. In this section we give some pictorial examples of the scaling functions obtained by applying (3.10)–(3.12) to the case of $m = 3$ and $N = 2$. We require

$$C_{00} + C_{03} + C_{30} + C_{33} = 1/2^2,$$
$$|C_{00}+C_{03}| < 1/2^{1+\ell},\ |C_{03}+C_{33}| < 1/2^{1+\ell},\ |C_{33}+C_{30}| < 1/2^{1+\ell},\ |C_{30}+C_{00}| < 1/2^{1+\ell},$$
$$|C_{00}| < 1/2^{\ell},\ |C_{03}| < 1/2^{\ell},\ |C_{30}| < 1/2^{\ell},\ |C_{33}| < 1/2^{\ell}.$$

The remaining C_K's are given by (3.10), which in two dimensions reads

$$C_{ij} = C_{00}\binom{m-1}{i}\binom{m-1}{j} + C_{0m}\binom{m-1}{i}\binom{m-1}{j-1}$$
$$+C_{m0}\binom{m-1}{i-1}\binom{m-1}{j} + C_{mm}\binom{m-1}{i-1}\binom{m-1}{j-1}.$$

In our example, $m = 3$, $(i, j) \in \{0, 1, 2, 3\}^2$, and the support of ϕ is $[0, 3]^2$. For a given set of coefficients, C_K, the supremum of all possible real values of ℓ will be shown by ℓ_c, the critical exponent. Then, for any $\ell < \ell_c$, the $[\ell]$th derivative of ϕ is Hölder continuous with exponent (at least) $\ell - [\ell]$.

If we choose $C_{00} = C_{03} = C_{30} = C_{33} = 1/16$, then we get the two-dimensional spline in Figure 1. This function fails to have continuous second derivatives at points in its support where one of the coordinates is an integer. The maximum integer value that we can use for ℓ in Theorem 3.3 is 1. Therefore, this function is $\mathcal{C}^1$. The critical exponent is $\ell_c = 2$.

If we choose $C_{00} = -0.075$, $C_{03} = C_{30} = 0.1$, and $C_{33} = 0.125$, then we get the graph in Figure 2. The maximum value that we can use for ℓ in Theorem 3.3 is 1. Therefore, this function is $\mathcal{C}^1$, despite appearances. The critical exponent is $\ell_c = -\log_2 0.45 = 1.152\ldots$.

If we choose $C_{00} = -0.5$, $C_{03} = C_{30} = 0.625$, and $C_{33} = -0.5$, then we get the graph in Figure 3. The maximum value that we can use for ℓ in Theorem 3.3 is 0. Therefore this function is only continuous. The critical exponent is $\ell_c = -\log_2 0.625 = 0.678\ldots$.

Appendix A. Some notes on binomial identities. Here we outline the proofs of some binomial identities used in this paper.

Note A.1. To verify that (2.12) is a solution of (2.11), one evaluates

$$\left(x\frac{d}{dx}\right)^q [x^b(1-x)^{m-1}] = \sum_{k=0}^{m}(-1)^k(k+b)^q\binom{m-1}{k}x^{k+b} \tag{A.1}$$

for $0 \le q \le m - 2$, and $b = 0$ or -1 at $x = 1$.

The fact that matrices given by (2.13) are inverses of each other follows from (A.1) for $q = b = 0$.

Note A.2. The matrices $\tilde{T}_0$ and $\tilde{T}_1$ and their divided differences have a particular zero structure and a simple formula for the entries of the first nonzero subdiagonal. The divided differences in question are polynomials such as $\tilde{T}_1 - \tilde{T}_0$, $\tilde{T}_1\tilde{T}_1 - 2\tilde{T}_1\tilde{T}_0 + \tilde{T}_0\tilde{T}_1$, etc., where the indices form an arithmetic sequence of binary numbers and the coefficients are the binomial numbers. This is discussed in the following theorem.

THEOREM A.1. *Consider a set of equidistant numbers $w_j = \theta + j\hbar$ for $j = 0, \ldots, \ell$, in the unit interval and with binary expansions that differ on the first k digits only.*

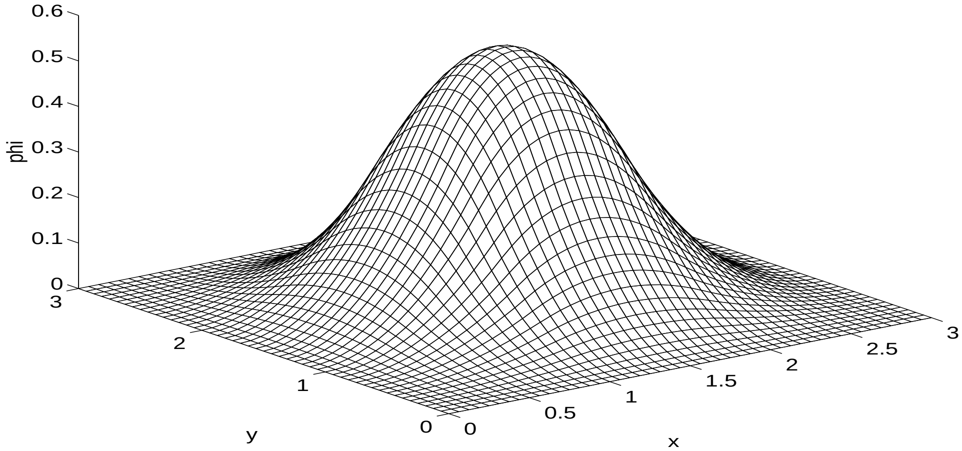

FIG. 3.1. *The scaling function for* $C_{00} = C_{03} = C_{30} = C_{33} = 1/16$*. Here,* $\ell_c = 2$*.*

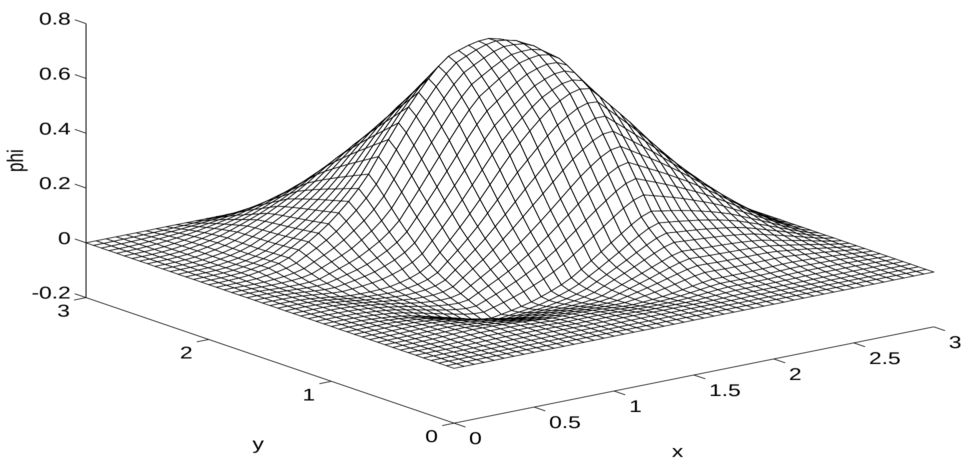

FIG. 3.2. *The scaling function for* $C_{00} = -0.075$*,* $C_{03} = C_{30} = 0.1$*,* $C_{33} = 0.125$*. Here,* $\ell_c = 1.152\ldots$*.*

Define

$$\tilde{\mathcal{D}} = \tilde{\mathcal{D}}(k, \ell, \hbar, \theta) = \frac{1}{\hbar^\ell} \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \tilde{P}_k(w_{\ell-i}). \tag{A.2}$$

Then

$$\tilde{\mathcal{D}}_{i,j} = 0 \qquad \text{if} \qquad i < m \qquad \text{and} \qquad i < j + \ell, \tag{A.3}$$

$$\tilde{\mathcal{D}}_{m,j} = 0 \qquad \text{if} \qquad c_0 = c_m \qquad \text{and} \qquad m < j + \ell, \tag{A.4}$$

$$\tilde{\mathcal{D}}_{i,i-\ell} = \ell! \binom{i-1}{\ell} 2^{k(m+\ell-i-1)} (c_0 + c_m)^k \qquad \text{for} \qquad \ell < i < m. \tag{A.5}$$

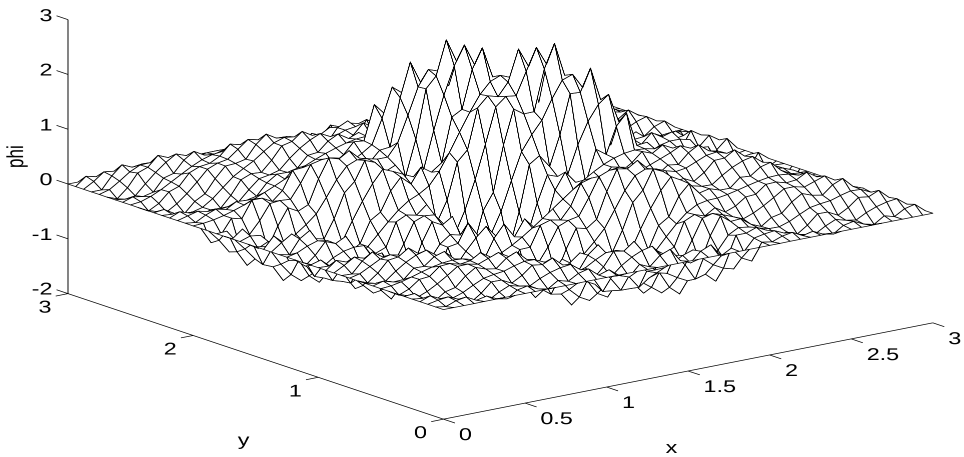

FIG. 3.3. *The scaling function for $C_{00} = -0.5$, $C_{03} = C_{30} = 0.625$, and $C_{33} = -0.5$. Here,* $\ell_c = 0.678\ldots$.

In particular, if c_k's satisfy the unit column sum rule, then $c_0 + c_k = 1/2^{m-2}$ and we have

$$\tilde{\mathcal{D}}_{i,i-\ell} = \ell!\binom{i-1}{\ell}2^{k(\ell+1-i)}. \tag{A.6}$$

The first entry of this list is

$$\tilde{\mathcal{D}}_{\ell+1,1} = \ell!. \tag{A.7}$$

Proof. We prove the theorem for $\tilde{T}_1$ and $\tilde{T}_0$. The general case is similar and can be shown by induction on ℓ. The proof rests on simple divided difference properties of polynomials. We establish that $S^{-1}T_0S$ and $S^{-1}T_1S$ are lower triangular, with diagonal entries given by (2.14). Define the matrices M_a for $a = 0, 1, 2$ by $(M_a)_{i,j} = \binom{m-1}{2i-j-a}$. We have $T_0 = c_0M_1 + c_mM_2$ and $T_1 = c_0M_0 + c_mM_1$. First, we show that $S^{-1}M_aS$ is lower triangular and determine its diagonal entries. The main step is to prove the following identity:

$$\sum_{1\le k,l\le m}(i-1)!\binom{m-k}{i-1}\binom{m-1}{2k-l-a}\binom{j-1}{m-l}\frac{(-1)^{l+j-m-1}}{(j-1)!} = 0 \tag{A.8}$$

for $j > i$ and $a = 0, 1, 2$.

A brief outline of the proof of (A.8) is as follows. First, we notice that for each fixed i the elements of row i of S^{-1} are the values of a polynomial of order $i-1$ in j. Then we will show that the same is true of $S^{-1}M_a$ (except that when $a = 0$, the last row is a polynomial of order $m-2$, and when $a = 2$, the last row is zero). Next we observe that for each j the elements of the column j of S are proportional to the coefficients of the divided difference scheme of order $j-1$. Therefore, when $j > i$, the product of row i of $S^{-1}M_a$ and column j of S is zero; hence, (A.8) follows.

Now we show that row i of $S^{-1}M_a$ is a polynomial of degree at most $i-1$. For a fixed i let $s(j) = S^{-1}_{ij}$. Obviously s is polynomial of degree $i-1$. Define g and h in terms of alternate values of $S^{-1}M_a$ on row i; that is,

$$\text{(A.9)}\qquad \sum_{1\le k\le m}(i-1)!\binom{m-k}{i-1}\binom{m-1}{2k-l-a}=\begin{cases} g(l) & \text{for } l \text{ odd},\\ h(l) & \text{for } l \text{ even}.\end{cases}$$

The alternate columns of S_{ij}^{-1} are identical up to a shift. Hence, for the j's with same parity, we get g as the same linear combination of translates of s. Therefore, g is a polynomial of degree $i-1$. A similar argument applies to h. We will show that for $i<m$ these two polynomials are identical. Define $f(x)=s(x/2)$, and notice that the leading term of f is $(-x/2)^{i-1}$. Assume $a=0$ (the cases for $a=1$ and $a=2$ can be treated similarly). Then we write (A.9) as

$$\text{(A.10)}\qquad \begin{aligned} g(x) &= \textstyle\sum_r \binom{m-1}{2r-1} f(x+2r-1),\\ h(x) &= \textstyle\sum_r \binom{m-1}{2r} f(x+2r). \end{aligned}$$

Suppose $f(x)=\sum_p a_p x^p$, where $a_p=0$ for $p<0$ or $p>i-1$. Then from binomial expansion of (A.10) we obtain

$$\text{(A.11)}\qquad \begin{aligned} g(x) &= \textstyle\sum_{p,q} a_p\binom{p}{q}x^{p-q}\sum_r\binom{m-1}{2r-1}(2r-1)^q,\\ h(x) &= \textstyle\sum_{p,q} a_p\binom{p}{q}x^{p-q}\sum_r\binom{m-1}{2r}(2r)^q. \end{aligned}$$

But from Note A.1 we have

$$\text{(A.12)}\qquad \sum_r\binom{m-1}{2r-1}(2r-1)^q=\sum_r\binom{m-1}{2r}(2r)^q \qquad \text{for} \qquad 0\le q\le m-2;$$

therefore, $g(x)$ and $h(x)$ are identical if their degree $i-1$ does not exceed $m-2$; that is, if $i<m$. In this case the leading term of the polynomial, from (2.19), is $2^{m-2}(-x/2)^{i-1}$. Now the entries on column j of S may be written as $[(-1)^{i+j}\binom{j-1}{m-i}]\times[(-1)^{m+1}/(j-1)!]$. The first part is the divided difference scheme of order $j-1$, and the second part is a constant. Therefore, the product of row i of $S^{-1}M_a$ and column j of S is zero when $i<j$. Hence, $S^{-1}M_aS$ is lower triangular. When $i=j<m$, then the product is 2^{m-i-1}. For $i=j=m$ we need to distinguish among three cases. When $a=0$, the last row of $S^{-1}M_a$ is $[m-1,1,0,0,\dots,0]$. The interpolating polynomial of these values is of degree $m-1$. Hence, its product with column m of S is zero. When $a=2$, then the last row itself is zero. When $a=1$, we get a nonzero contribution, i.e., 1. This explains the particular form of the eigenvalues in (2.14). ∎

Acknowledgments. I am grateful to Professor Gilbert Strang for interesting me in this topic, to the referees for pointing out several corrections, and to Professor Christopher Heil for valuable suggestions.

REFERENCES

[1] M. A. Berger and Y. Wang, *Bounded semi-groups of matrices*, Linear Algebra Appl., 166 (1992), pp. 21–27.

[2] A. S. Cavaretta, W. Dahmen, and C. A. Micchelli, *Stationary subdivision*, Mem. Amer. Math. Soc., 93 (1991), pp. 1–186.

[3] J. E. Cohen, *Subadditivity, generalized products of random matrices and operations research*, SIAM Rev., 30 (1988), pp. 69–86.

[4] D. Colella and C. Heil, *The characterization of continuous, four-coefficient scaling functions and wavelets*, IEEE Trans. Inform. Theory, Special Issue on Wavelet Transforms and Multiresolution Signal Analysis, 38 (1992), pp. 876–881.

[5] D. Colella and C. Heil, *Characterizations of scaling functions, continuous solutions*, SIAM J. Matrix Anal. Appl., 15 (1994), no. 2, pp. 496–518.

[6] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math., 49 (1988), pp. 909–996.

[7] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.

[8] I. Daubechies and J. Lagarias, *Two-scale difference equations,* I. *Existence and global regularity of solutions*, SIAM J. Math. Anal., 22 (1991), pp. 1388–1410.

[9] I. Daubechies and J. Lagarias, *Two-scale difference equations,* II. *Local regularity, infinite products of matrices and fractals*, SIAM J. Math. Anal., 23 (1992), pp. 1031–1079.

[10] I. Daubechies and J. Lagarias, *Sets of matrices all infinite products of which converge*, Linear Algebra Appl., 161 (1992), pp. 227–263.

[11] C. Heil and G. Strang, *Continuity of the joint spectral radius: Application to wavelets*, in Linear Algebra for Signal Processing, IMA Vol. Math. Appl. 69, A. Bojanczyk and G. Cybenko, eds., Springer-Verlag, New York, 1995. pp. 51–61,

[12] J. C. Lagarias and Y. Wang, *The finiteness conjecture for the generalized spectral radius of a set of matrices*, Linear Algebra Appl., 214 (1995), pp. 17–42.

[13] C. A. Micchelli and H. Prautzsch, *Refinement and subdivision for spaces of integer translates of a compactly supported function*, in Numerical Analysis 1987, Pitman Res. Notes Math. Ser. 170, D. F. Griffiths and G. A. Watson, eds., Longman Sci. Tech., Harlow, UK, 1988, pp. 192–222.

[14] C. A. Micchelli and H. Prautzsch, *Uniform refinement of curves,* Linear Algebra Appl., 114/115 (1989), pp. 841–870.

[15] M. Maesumi, *Optimal unit ball for joint spectral radius, an example from four-coefficient MRA*, in Approximation Theory VIII, Vol. 2: Wavelets and Multilevel Approximation, Proceedings of the Eighth International Conference on Approximation Theory, C. K. Chui, and L. L. Schumaker, eds, World Scientific, Singapore, pp. 267–274.

[16] G. C. Rota and G. Strang, *A note on the joint spectral radius*, Konink. Nederl. Akad. Wetensch. Proc. A, 63 (1960), pp. 379–381.

[17] G. Strang, *Wavelets and dilation equations: A brief introduction*, SIAM Rev., 31 (1989), pp. 614–627.

[18] G. Strang, *Wavelet transforms versus Fourier transforms*, Bull. Amer. Math. Soc. (N.S.), 28 (1993), pp. 288–305.