**Multiple Integration by Parts**

Here is an approach to this rather confusing topic, with a slightly different notation.
Let's define 'down' as

$$f^{\downarrow} = f'$$

and 'up' as

$$f^{\uparrow} = \int f$$

and similarly 'double down' as $f^{\downarrow\downarrow} = f''$ and 'double up' as $f^{\uparrow\uparrow} = \int\int f$ and so on.
Now to do one round of integration by parts use

$$\int fg = fg^{\uparrow} - \int (f^{\downarrow} g^{\uparrow})$$

To do two rounds use

$$\int fg = fg^{\uparrow} - f^{\downarrow} g^{\uparrow\uparrow} + \int (f^{\downarrow\downarrow} g^{\uparrow\uparrow})$$

You notice in the integrals the number of ups and downs are equal, and the integral itself counts as one up. With respect to other terms $f$ starts as $f$ and then goes 'down' while $g$ starts to go 'up' immediately. So in each term the number of 'up' moves is one more than 'down' moves.
To do three rounds use

$$\int fg = fg^{\uparrow} - f^{\downarrow} g^{\uparrow\uparrow} + f^{\downarrow\downarrow} g^{\uparrow\uparrow\uparrow} - \int (f^{\downarrow\downarrow\downarrow} g^{\uparrow\uparrow\uparrow}).$$

For example let us do $\int x^3 e^{2x} dx$. This is simple because $f$ is a polynomial so its 'down' is a simpler polynomial. we notice that the fourth (and subsequent) derivatives of $x^3$ will be zero so we go that far and get a convenient zero in the last integral
$\int x^3 e^{2x} = (x^3)(e^{2x})^{\uparrow} - (x^3)^{\downarrow}(e^{2x})^{\uparrow\uparrow} + (x^3)^{\downarrow\downarrow}(e^{2x})^{\uparrow\uparrow\uparrow} - (x^3)^{\downarrow\downarrow\downarrow}(e^{2x})^{\uparrow\uparrow\uparrow\uparrow} + \int (x^3)^{\downarrow\downarrow\downarrow\downarrow}(e^{2x})^{\uparrow\uparrow\uparrow\uparrow}$
$\int x^3 e^{2x} = (x^3)(e^{2x}/2) - (3x^2)(e^{2x}/4) + (6x)(e^{2x}/8) - (6)(e^{2x}/16) + C$
Now, in the next example, we deal with a tricky integral $\int e^{ax} \cos bx$. Here we go so far on the right until the integral looks like the original one, just differing by a constant multiple. At which point we stop and bring the right integral to the left, hence completing the circle so to speak. Here is how it goes: for cosine two rounds of either up or down brings us back to cosine so we use
$\int e^{ax} \cos bx = (e^{ax})(\cos bx)^{\uparrow} - (e^{ax})^{\downarrow}(\cos bx)^{\uparrow\uparrow} + \int (e^{ax})^{\downarrow\downarrow}(\cos bx)^{\uparrow\uparrow}$
$\int e^{ax} \cos bx = (e^{ax})(\sin bx/b) - (ae^{ax})(-\cos bx/b^2) + \int (a^2 e^{ax})(-\cos bx/b^2)$
Now you bring the last integral to the left and factor it to get $(1 + a^2/b^2) \int e^{ax} \cos bx = 1/b e^{ax} \sin bx + a/b^2 e^{ax} \cos bx + K$. Now we clean up and we have

$$\int e^{ax} \cos bx = \frac{e^{ax}(b \sin bx + a \cos bx)}{(a^2 + b^2)} + C$$