34. In economics, the graph in Figure 2.3 is called the **Phillips curve**, after A. W. Phillips, a New Zealander associated with the London School of Economics. Until Phillips published his ideas in the 1950s, many economists believed that unemployment and inflation were linearly related. Read an article on the Phillips curve (the source cited with Example 1.1 would be a good place to start) and write a paragraph on the nature of unemployment in the U.S. economy.

35. Find the slope of the line that is tangent to the graph of the function $f(x) = \sqrt{x^2 + 2x - \sqrt{3x}}$ at the point where $x = 3.85$ by filling in the following chart. Record all calculations using five decimal places.

<table>
<thead>
<tr>
<th>$\Delta x$</th>
<th>$-0.02$</th>
<th>$-0.01$</th>
<th>$-0.001$</th>
<th>$0$</th>
<th>$0.001$</th>
<th>$0.01$</th>
<th>$0.02$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + \Delta x$</td>
<td></td>
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<tr>
<td>$f(x)$</td>
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<tr>
<td>$f(x + \Delta x)$</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$\frac{f(x + \Delta x) - f(x)}{\Delta x}$</td>
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</tr>
</tbody>
</table>

36. Find the $x$ values at which the peaks and valleys of the graph of $y = 2x^3 - 0.8x^2 + 4$ occur. Use four decimal places.

37. Show that $f(x) = \frac{|x^2 - 1|}{x - 1}$ is not differentiable at $x = 1$.

---

### Techniques of Differentiation

**Chapter 2 - Section 2**

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**The Constant Rule**

For any constant $c$,

$$\frac{d}{dx}(c) = 0$$

That is, the derivative of a constant is zero.

You can see this by considering the graph of a constant function $f(x) = c$, which is a horizontal line (see Figure 2.7). Since the slope of such a line is 0 at all its points, it follows that $f'(x) = 0$. Here is a proof using the limit definition:
The next rule is one of the most useful because it tells you how to take the derivative of any power function \( f(x) = x^n \). Note that the rule applies not only to functions like \( f(x) = x^5 \) but also to those such as \( g(x) = \sqrt[4]{x^4} = x^{4/5} \) and \( h(x) = \frac{1}{x^2} = x^{-2} \). Functions such as \( x^{\sqrt{3}} \) can also be differentiated using the power rule, but we do not define what is meant by such functions until Chapter 4.

**The Power Rule** For any real number \( n \),

\[
\frac{d}{dx} (x^n) = nx^{n-1}
\]

In words, to find the derivative of \( x^n \), reduce the exponent \( n \) of \( x \) by 1 and multiply by the original exponent.

**Example 2.1**

\[
\frac{d}{dx} (-15) = 0
\]

The next rule is one of the most useful because it tells you how to take the derivative of any power function \( f(x) = x^n \). Note that the rule applies not only to functions like \( f(x) = x^5 \) but also to those such as \( g(x) = \sqrt[4]{x^4} = x^{4/5} \) and \( h(x) = \frac{1}{x^2} = x^{-2} \). Functions such as \( x^{\sqrt{3}} \) can also be differentiated using the power rule, but we do not define what is meant by such functions until Chapter 4.

**Example 2.2**

\[
\frac{d}{dx} (x^7) = 7x^{7-1} = 7x^6
\]
\[
\frac{d}{dx} (\sqrt[3]{x^3}) = \frac{d}{dx} (x^{3/3}) = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3}
\]
\[
\frac{d}{dx} \left( \frac{1}{x^3} \right) = \frac{d}{dx} (x^{-5}) = -5x^{-5-1} = -5x^{-6}
\]

We essentially proved the power rule for \( f(x) = x^3 \) in Example 1.2 in the last section, and for \( f(x) = \sqrt{x} = x^{-1/2} \) in Example 1.4. For \( n = -2 \), the power rule says that the function \( F(x) = x^{-2} \) has the derivative \( F'(x) = -2x^{-3} \). Here is a proof of the power rule for this function:
A general proof for the power rule in the case where \( n \) is a positive integer is outlined in Problem 63. The case where \( n \) is a negative integer is handled in Section 3 of this chapter, and that where \( n \) is a rational number \( \left( n = \frac{r}{s} \right) \), in Section 7 of this chapter.

The constant rule and power rule provide simple formulas for finding derivatives of a class of important functions, but to differentiate more complicated expressions, we need to know how to manipulate derivatives algebraically. The next two rules tell us that derivatives of multiples and sums of functions are multiples and sums of the corresponding derivatives.

\[
F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = \lim_{h \to 0} \frac{[1/(x + h)^2] - [1/x^2]}{h}
\]

\[
= \lim_{h \to 0} \frac{x^2 - (x^2 + 2hx + h^2)}{hx^2(x + h)^2} = \lim_{h \to 0} \frac{-2hx - h^2}{hx^2(x + h)^2}
\]

\[
= \lim_{h \to 0} \frac{-2x - h}{x^2(x + h)^2} = -\frac{2x}{x^4}
\]

\[
= -2x^{-3}
\]

A general proof for the power rule in the case where \( n \) is a positive integer is outlined in Problem 63. The case where \( n \) is a negative integer is handled in Section 3 of this chapter, and that where \( n \) is a rational number \( \left( n = \frac{r}{s} \right) \), in Section 7 of this chapter.

The constant rule and power rule provide simple formulas for finding derivatives of a class of important functions, but to differentiate more complicated expressions, we need to know how to manipulate derivatives algebraically. The next two rules tell us that derivatives of multiples and sums of functions are multiples and sums of the corresponding derivatives.

**The Constant Multiple Rule**  
If \( c \) is a constant and \( f(x) \) is differentiable, then so is \( cf(x) \) and

\[
\frac{d}{dx} (cf(x)) = c \frac{df}{dx}
\]

That is, the derivative of a multiple is the multiple of the derivative.

**EXAMPLE 2.3**

\[
\frac{d}{dx} (3x^4) = 3 \frac{d}{dx} (x^4) = 3(4x^3) = 12x^3
\]

\[
\frac{d}{dx} \left( -7 \frac{1}{\sqrt{x}} \right) = \frac{d}{dx} (-7x^{-1/2}) = -7 \left( -\frac{1}{2} \frac{1}{x^{3/2}} \right) = \frac{7}{2} x^{-3/2}
\]

**The Sum Rule**  
If \( f(x) \) and \( g(x) \) are differentiable, then so is the sum \( S(x) = f(x) + g(x) \) and \( S'(x) = f'(x) + g'(x) \); that is,

\[
\frac{d}{dx} [f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}
\]

In words, the derivative of a sum is the sum of the separate derivatives.
By combining the power rule, the constant multiple rule, and the sum rule, you can differentiate any polynomial. Here is an example.

**Example 2.4**

\[
\frac{d}{dx} (x^{-2} + 7) = \frac{d}{dx} (x^{-2}) + \frac{d}{dx} (7) = -2x^{-3} + 0 = -2x^{-3}
\]

\[
\frac{d}{dx} [2x^5 - 3x^{-7}] = 2 \frac{d}{dx} (x^5) - 3 \frac{d}{dx} (x^{-7}) = 2(5x^4) - 3(-7x^{-8})
\]

\[
= 10x^4 + 21x^{-8}
\]

By combining the power rule, the constant multiple rule, and the sum rule, you can differentiate any polynomial. Here is an example.

**Example 2.5**

Differentiate the polynomial \(y = 5x^3 - 4x^2 + 12x - 8\).

**Solution**

Differentiate this sum term by term to get

\[
\frac{dy}{dx} = \frac{d}{dx} (5x^3) + \frac{d}{dx} (-4x^2) + \frac{d}{dx} (12x) + \frac{d}{dx} (-8)
\]

\[
= 15x^2 - 8x^1 + 12x^0 + 0
\]

\[
= 15x^2 - 8x + 12
\]

**Example 2.6**

It is estimated that \(x\) months from now, the population of a certain community will be \(P(x) = x^2 + 20x + 8,000\).

(a) At what rate will the population be changing with respect to time 15 months from now?

(b) By how much will the population actually change during the 16th month?

**Solution**

(a) The rate of change of the population with respect to time is the derivative of the population function. That is,

\[
\text{Rate of change} = P'(x) = 2x + 20
\]

The rate of change of the population 15 months from now will be

\[
P'(15) = 2(15) + 20 = 50 \text{ people per month}
\]
(b) The actual change in the population during the 16th month is the difference between the population at the end of 16 months and the population at the end of 15 months. That is,

\[
\text{Change in population} = P(16) - P(15) = 8,576 - 8,525 = 51 \text{ people}
\]

In Example 2.6, the actual change in population during the 16th month in part (b) differs from the monthly rate of change at the beginning of the month in part (a) because the rate varies during the month. The instantaneous rate in part (a) can be thought of as the change in population that would occur during the 16th month if the rate of change of population were to remain constant.

**Percentage Rate of Change**

In many practical situations, the instantaneous rate of change of a quantity is not as significant as its **percentage rate of change**, which is defined by the formula

\[
\left( \frac{\text{percentage rate of change}}{\text{of change of } Q} \right) = 100 \frac{\text{rate of change of } Q}{\text{size of } Q} = \frac{100Q'(x)}{Q(x)}
\]

For example, a yearly rate of change of $500 in salary would not mean a great deal to a person earning $5 million. In fact, it amounts to a percentage rate of change of only

\[
\frac{100(500)}{5,000,000} = 0.01\%
\]

However, to a person earning only $2,000 per year, an annual $500 rate of increase amounts to a percentage rate of

\[
\frac{100(500)}{2,000} = 25\%
\]

Here are two examples involving percentage rate of change.

**Example 2.7**

The gross domestic product (GDP) of a certain country was \( N(t) = t^2 + 5t + 106 \) billion dollars \( t \) years after 1990.
(a) At what rate was the GDP changing with respect to time in 1998?
(b) At what percentage rate was the GDP changing with respect to time in 1998?

Solution
(a) The rate of change of the GDP is the derivative \( N'(t) = 2t + 5 \). The rate of change in 1998 was \( N'(8) = 2(8) + 5 = 21 \) billion dollars per year.
(b) The percentage rate of change of the GDP in 1998 was
\[
100 \frac{N'(8)}{N(8)} = 100 \frac{21}{210} = 10\% \text{ per year}
\]

Experiment 2.8
Experiments indicate that the biomass \( Q(t) \) of a fish species in a given area of the ocean changes at the rate
\[
\frac{dQ}{dt} = rQ \left( 1 - \frac{Q}{a} \right)
\]
where \( r \) is the natural growth rate of the species and \( a \) is a constant.* Find the percentage rate of growth of the species. What if \( Q(t) > a \)?

Solution
The percentage rate of change of \( Q(t) \) is
\[
100 \frac{Q'(t)}{Q(t)} = 100 \frac{rQ \left( 1 - \frac{Q}{a} \right)}{Q} = 100r \left( 1 - \frac{Q}{a} \right)
\]
Notice that the percentage rate decreases as \( Q \) increases and becomes zero when \( Q = a \). If \( Q > a \), the percentage rate is negative, which means the biomass is actually decreasing.

Rectilinear Motion
Rectilinear motion is motion of an object along a straight line. For example, the motion of a rocket early in its flight can be regarded as rectilinear. When studying rectilinear motion, you may assume that the object is moving along a coordinate axis.

The position of the object is a function of time $t$ and is often denoted by $s(t)$. The rate of change of position with respect to time is the object’s **velocity**, and the rate of change of velocity with respect to time is its **acceleration** $a(t)$; that is, $v(t) = s'(t)$ and $a(t) = v'(t)$. The object is said to be **stationary** when $v(t) = 0$.

### Example 2.9

An object moves along a line in such a way that its position at time $t$ is $s(t) = t^3 - 6t^2 + 9t + 5$.

(a) Find the velocity and acceleration of the object at time $t$.

(b) When is the object stationary?

**Solution**

(a) The velocity is

$$v(t) = s'(t) = 3t^2 - 6(2t) + 9 + 0 = 3t^2 - 12t + 9$$

and the acceleration is

$$a(t) = v'(t) = 3(2t) - 12 + 0 = 6t - 12$$

(b) The object is stationary when

$$v(t) = 3t^2 - 12t + 9 = 0$$

that is, when $t = 1$, $t = 3$.

---

**The Motion of a Projectile**

An important example of rectilinear motion is the motion of a projectile. Suppose an object is projected (e.g., thrown, fired, or dropped) vertically in such a way that the only acceleration acting on the object is the constant downward acceleration $g$ due to gravity. Near sea level, $g$ is approximately 32 ft/sec$^2$ (or 9.8 m/sec$^2$). It can be shown that at time $t$, the height of the object is given by the formula

$$H(t) = \frac{-1}{2} gt^2 + V_0t + H_0$$

where $H_0$ and $V_0$ are the initial height and velocity of the object, respectively. Here is an example using this formula.

### Example 2.10

Suppose a person standing at the top of a building 112 feet high throws a ball vertically upward with an initial velocity of 96 ft/sec (see Figure 2.8).
(a) Find the ball’s height and velocity at time $t$.
(b) When does the ball hit the ground and what is its impact velocity?
(c) When is the velocity 0? What is the significance of this time?
(d) How far does the ball travel during its flight?

Solution

(a) Since $g = 32$, $V_0 = 96$, and $H_0 = 112$, the height of the ball above the ground at time $t$ is

$$H(t) = -16t^2 + 96t + 112 \text{ feet}$$

The velocity at time $t$ is

$$v(t) = \frac{dH}{dt} = -32t + 96 \text{ (ft/sec)}$$

(b) The ball hits the ground when $H = 0$. Solve the equation $-16t^2 + 96t + 112 = 0$ to find that this occurs when $t = 7$ and $t = -1$ (verify). Disregarding the negative time $t = -1$, which is not meaningful in this context, conclude that impact occurs when $t = 7$ seconds and that the impact velocity is

$$v(7) = -32(7) + 96 = -128 \text{ ft/sec}$$

(The negative sign means the ball is coming down at the moment of impact.)

(c) The velocity is zero when $v(t) = -32t + 96 = 0$, which occurs when $t = 3$ seconds. For $t < 3$, the velocity is positive and the ball is rising, and for $t > 3$, the ball is falling (see Figure 2.8). Thus, the ball is at its highest point when $t = 3$ seconds.

(d) The ball starts at $H(0) = 112$ feet and rises to a maximum height of $H(3) = 256$ feet before falling to the ground. Thus,

$$\text{Total distance traveled} = (256 - 112) + 256 = 400 \text{ feet}$$

FIGURE 2.8  The motion of a ball thrown upward from the top of a building.
In Problems 1 through 16, differentiate the given function. Do as much of the computation as possible in your head and simplify your answers.

1. \( y = x^{-4} \)
2. \( y = x^{7/3} \)
3. \( y = \frac{9}{\sqrt{t}} \)
4. \( y = \frac{3}{2t^2} \)
5. \( y = x^2 + 2x + 3 \)
6. \( y = 3x^5 - 4x^3 + 9x - 6 \)
7. \( f(x) = x^9 - 5x^8 + x + 12 \)
8. \( f(x) = \frac{1}{4}x^8 - \frac{1}{2}x^6 - x + 2 \)
9. \( y = \frac{1}{t} + \frac{1}{t^2} - \frac{1}{\sqrt{t}} \)
10. \( y = \frac{3}{x} - \frac{2}{x^2} + \frac{2}{3x^3} \)
11. \( f(x) = \sqrt[3]{x} + \frac{1}{\sqrt[3]{x^3}} \)
12. \( f(x) = 2\sqrt{t} + \frac{4}{\sqrt{t}} - \sqrt{2} \)
13. \( y = -\frac{x^2}{16} + \frac{2}{x} - x^{3/2} + \frac{1}{3x^2} + \frac{x}{3} \)
14. \( y = -\frac{2}{x^2} + x^{2/3} + \frac{1}{2\sqrt[3]{x}} + \frac{x^2}{4} + \sqrt[5]{x} + \frac{x + 2}{3} \)
15. \( y = x^2(x^3 - 6x + 7) \)  [Hint: Multiply first.]
16. \( y = \frac{x^5 - 4x^2}{x^3} \)  [Hint: Divide.]

In Problems 17 through 20 find the equation of the line that is tangent to the graph of the given function at the specified point.

17. \( y = x^5 - 3x^3 - 5x + 2; (1, -5) \)
18. \( y = -x^3 - 5x^2 + 3x - 1; (-1, -8) \)
19. \( y = \sqrt{x^3} - x^2 + \frac{16}{x^2}; (4, -7) \)
20. \( y = 1 - \frac{1}{x} + \frac{2}{\sqrt{x}}; \left(4, \frac{7}{4}\right) \)

In Problems 21 through 24 find the equation of the line that is tangent to the graph of the given function at the point \((x, f(x))\) for the specified value of \(x\).

21. \( f(x) = x^4 - 3x^3 + 2x^2 - 6; x = 2 \)
22. \( f(x) = x^3 + \sqrt{x}; x = 4 \)
23. \( f(x) = x - \frac{1}{x^2}; x = 1 \)
24. \( f(x) = x(\sqrt{x} - 1); x = 4 \)
In Problems 25 through 28 find the rate of change of the given function \( f(x) \) with respect to \( x \) for the prescribed value of \( x \).

25. \( f(x) = x^3 - 3x + 5; \ x = 2 \)

26. \( f(x) = \sqrt{x} + 5x; \ x = 4 \)

27. \( f(x) = x - \sqrt{x} + \frac{1}{x}; \ x = 1 \)

28. \( f(x) = \frac{x + \sqrt{x}}{x\sqrt{x}}; \ x = 1 \)

29. Sketch the graph of the function \( f(x) = x^2 - 4x - 5 \) and use calculus to determine its lowest point.

30. Sketch the graph of the function \( f(x) = 3 - 2x - x^2 \) and use calculus to determine its highest point.

31. Find numbers \( a \) and \( b \) such that the lowest point on the graph of the function \( f(x) = ax^2 + bx \) is \( (3, -8) \).

32. Find numbers \( a, b, \) and \( c \) such that the graph of the function \( f(x) = ax^2 + bx + c \) will have \( x \) intercepts at \((0, 0)\) and \((5, 0)\), and a tangent with slope 1 when \( x = 2 \).

33. Find the equations of all of the tangents to the graph of the function \( f(x) = x^2 - 4x + 25 \)
that pass through the origin \((0, 0)\).

34. Find all the points \((x, y)\) on the graph of the function \( y = 4x^2 \) with the property that the tangent to the graph at \((x, y)\) passes through the point \((2, 0)\).

CONSUMER EXPENDITURE

35. The consumer demand for a certain commodity is \( D(p) = -200p + 12,000 \) units per month when the market price is \( p \) dollars per unit.

(a) Express consumers’ total monthly expenditure for the commodity as a function of \( p \) and draw the graph.

(b) Use calculus to determine the market price for which the consumer expenditure is greatest.

LANDSCAPING

36. A landscaper wishes to enclose a rectangular flower garden with 20 feet of fencing. Use calculus to find the largest area that can be enclosed by such a garden.

NEWSPAPER CIRCULATION

37. It is estimated that \( t \) years from now, the circulation of a local newspaper will be \( C(t) = 100t^2 + 400t + 5,000 \).

(a) Derive an expression for the rate at which the circulation will be changing with respect to time \( t \) years from now.

(b) At what rate will the circulation be changing with respect to time 5 years from now? Will the circulation be increasing or decreasing at that time?

(c) By how much will the circulation actually change during the sixth year?

AIR POLLUTION

38. An environmental study of a certain suburban community suggests that \( t \) years from now, the average level of carbon monoxide in the air will be \( Q(t) = 0.05t^2 + 0.1t + 3.4 \) parts per million.

(a) At what rate will the carbon monoxide level be changing with respect to time 1 year from now?
(b) By how much will the carbon monoxide level change this year?
(c) By how much will the carbon monoxide level change over the next 2 years?

WORKER EFFICIENCY  39. An efficiency study of the morning shift at a certain factory indicates that an average worker who arrives on the job at 8:00 A.M. will have assembled

\[ f(x) = -x^3 + 6x^2 + 15x \]

transistor radios \( x \) hours later.

(a) Derive a formula for the rate at which the worker will be assembling radios after \( x \) hours.
(b) At what rate will the worker be assembling radios at 9:00 A.M.?
(c) How many radios will the worker actually assemble between 9:00 A.M. and 10:00 A.M.?

EDUCATIONAL TESTING  40. It is estimated that \( x \) years from now, the average SAT mathematics score of the incoming students at an eastern liberal arts college will be \( f(x) = -6x + 582 \).

(a) Derive an expression for the rate at which the average SAT score will be changing with respect to time \( x \) years from now.
(b) What is the significance of the fact that the expression in part (a) is a constant? What is the significance of the fact that the constant in part (a) is negative?

41. Two cars leave an intersection at the same time. One travels east at a constant speed of 60 kilometers per hour, while the other goes north at a constant speed of 80 kilometers per hour. Find an expression for the rate at which the distance between the cars is changing with respect to time.

42. Find the percentage rate of change in the function \( f(t) = 3t^2 - 7t + 5 \) with respect to \( t \), when \( t = 2 \).

POPULATION GROWTH  43. It is projected that \( x \) months from now, the population of a certain town will be \( P(x) = 2x + 4x^{3/2} + 5,000 \).

(a) At what rate will the population be changing with respect to time 9 months from now?
(b) At what percentage rate will the population be changing with respect to time 9 months from now?

ANNUAL EARNINGS  44. The gross annual earnings of a certain company were \( A(t) = 0.1t^2 + 10t + 20 \) thousand dollars \( t \) years after its formation in 1998.

(a) At what rate were the gross annual earnings of the company growing with respect to time in 2002?
(b) At what percentage rate were the gross annual earnings growing with respect to time in 2002?

PROPERTY TAX  45. Records indicate that \( x \) years after 1994, the average property tax on a three-bedroom home in a certain community was \( T(x) = 20x^2 + 40x + 600 \) dollars.

(a) At what rate was the property tax increasing with respect to time in 2000?
(b) At what percentage rate was the property tax increasing with respect to time in 2000?

**POPULATION GROWTH** 46. It is estimated that \( t \) years from now, the population of a certain town will be 
\[ P(t) = t^2 + 200t + 10,000. \]
(a) Express the percentage rate of change of the population as a function of \( t \), simplify this function algebraically, and draw its graph.
(b) What will happen to the percentage rate of change of the population in the long run (that is, as \( t \) grows very large)?

**SALARY INCREASES** 47. Your starting salary will be $24,000, and you will get a raise of $2,000 each year after your first year.
(a) Express the percentage rate of change of your salary as a function of time and draw the graph.
(b) At what percentage rate will your salary be increasing after 1 year?
(c) What will happen to the percentage rate of change of your salary in the long run?

**GROSS DOMESTIC PRODUCT** 48. The gross domestic product of a certain country is growing at a constant rate. In 1990 the GDP was 125 billion dollars, and in 1998 it was 155 billion dollars. If this trend continues, at what percentage rate will the GDP be growing in 2010?

**RECTILINEAR MOTION** In Problems 49 through 54, \( s(t) \) is the position of a particle moving along a straight line at time \( t \).
(a) Find the velocity and acceleration of the particle.
(b) Find all times in the given interval when the particle is stationary.

49. \( s(t) = t^2 - 2t + 6 \) for \( 0 \leq t \leq 2 \)
50. \( s(t) = 3t^2 + 2t - 5 \) for \( 0 \leq t \leq 1 \)
51. \( s(t) = t^3 - 9t^2 + 15t + 25 \) for \( 0 \leq t \leq 6 \)
52. \( s(t) = t^4 - 4t^3 + 8t \) for \( 0 \leq t \leq 4 \)
53. \( s(t) = 2t^4 + 3t^2 - 36t + 40 \) for \( 0 \leq t \leq 3 \)
54. \( s(t) = t^4 - 9t^3 + 15t + 25 \) for \( 0 \leq t \leq 6 \)

**MOTION OF A PROJECTILE** 55. A stone is dropped from a height of 144 feet.
(a) When will the stone hit the ground?
(b) With what velocity does it hit the ground?

**MOTION OF A PROJECTILE** 56. You are standing on the top of a building and throw a ball vertically upward. After 2 seconds, the ball passes you on the way down, and 2 seconds after that, it hits the ground below.
(a) What is the initial velocity of the ball?
(b) How high is the building?
(c) What is the velocity of the ball when it passes you on the way down?
(d) What is the velocity of the ball as it hits the ground?
57. Our friend, the spy who escaped from the diamond smugglers in Chapter 1 (Problem 38 of Section 4), is on a secret mission in space. An encounter with an enemy agent leaves him with a mild concussion and temporary amnesia. Fortunately, he has a book that gives the formula for the motion of a projectile and the values of $g$ for various heavenly bodies (32 ft/sec$^2$ on earth, 5.5 ft/sec$^2$ on the moon, 12 ft/sec$^2$ on Mars, and 28 ft/sec$^2$ on Venus). To deduce his whereabouts, he throws a rock vertically upward (from ground level) and notes that it reaches a maximum height of 37.5 ft and hits the ground 5 seconds after it leaves his hand. Where is he?

58. The graph below shows the pattern of unemployment (as a percentage of the workforce) for the time period 1973–1991.

(a) During what year was the percentage of unemployment the greatest for this period? When was it the least?

(b) Use the graph to estimate the rate of change of the percentage of unemployment during the election years of 1980 and 1988.

(c) Read an article on unemployment. Find data relating to unemployment rates by age, race, and sex for the period 1991–1999.* Then write a paragraph comparing and contrasting the unemployment pattern of the decade 1990–1999 with the patterns of the previous two decades (1970–1989).

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**PROBLEM 58**

*Source: Adapted from Rural Conditions and Trends, Economic Research Service, U.S. Dept. of Agriculture, Vol. 3, No. 3 (Fall 1992).*

59. Use calculus to prove that if $y$ is a linear function of $x$, the rate of change of $y$ with respect to $x$ is constant.

60. If \( y \) is a linear function of \( x \), what will happen to the percentage rate of change of \( y \) with respect to \( x \) as \( x \) increases without bound? Explain.

**MANUFACTURING COST**

61. Suppose the total manufacturing cost \( C \) at a certain factory is a function of the number \( q \) of units produced, which in turn is a function of the number \( t \) of hours during which the factory has been operating.

(a) What quantity is represented by the derivative \( \frac{dC}{dq} \)? In what units is this quantity measured?

(b) What quantity is represented by the derivative \( \frac{dq}{dt} \)? In what units is this quantity measured?

(c) What quantity is represented by the product \( \frac{dC}{dq} \frac{dq}{dt} \)? In what units is this quantity measured?

62. Prove the sum rule for derivatives. \( \text{[Hint: Note that the difference quotient for } f + g \text{ can be written as} \) 

\[
\frac{(f + g)(x + h) - (f + g)(x)}{h} = \frac{[f(x + h) + g(x + h)] - [f(x) + g(x)]}{h}
\]

63. (a) If \( f(x) = x^4 \), show that \( \frac{f(x + h) - f(x)}{h} = 4x^3 + 6x^2h + 4xh^2 + h^3 \)

(b) If \( f(x) = x^n \) for positive integer \( n \), show that

\[
\frac{f(x + h) - f(x)}{h} = nx^{n-1} + \frac{n(n - 1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}
\]

(c) Use the result in part (b) in the definition of the derivative to prove the power rule:

\[
\frac{d}{dx}(x^n) = nx^{n-1}
\]

Based on your experience with the multiple and sum rules in Section 2 of this chapter, you may think that the derivative of a product of functions is the product of separate derivatives, but it is easy to see that this conjecture is false. For instance, if \( f(x) = x^2 \) and \( g(x) = x^3 \), then \( f'(x) = 2x \) and \( g'(x) = 3x^2 \), so

\[
f'(x)g'(x) = (2x)(3x^2) = 6x^3
\]

while \( f(x)g(x) = x^2x^3 = x^5 \) and

\[
[f(x)g(x)]' = (x^5)' = 5x^4
\]