

In Problems 50 through 53, use the method of Lagrange multipliers to find the indicated maximum or minimum. You will need to use the graphing utility or the solve application on your calculator.



50. Maximize  $f(x, y) = e^{x+y} - x \ln\left(\frac{y}{x}\right)$  subject to  $x + y = 4$ .



51. Minimize  $f(x, y) = \ln(x + 2y)$  subject to  $xy + y = 5$ .

52. Minimize  $f(x, y) = \frac{1}{x^2} + \frac{3}{xy} + \frac{1}{y^2}$  subject to  $x + 2y = 7$ .



53. Maximize  $f(x, y) = xe^{x^2-y}$  subject to  $x^2 + 2y^2 = 1$ .

# 5

## Double Integrals over Rectangular Regions

In Chapters 5 and 6, you integrated a function of one variable  $f(x)$  by reversing the process of differentiation, and a similar procedure can be used to integrate a function of two variables  $f(x, y)$ . However, since two variables are involved, we shall integrate  $f(x, y)$  by holding one variable fixed and integrating with respect to the other.

For instance, to evaluate the partial integral  $\int_1^2 xy^2 dx$  you would integrate with respect to  $x$ , using the fundamental theorem of calculus with  $y$  held constant:

$$\begin{aligned}\int_1^2 xy^2 dx &= \frac{1}{2}x^2 y^2 \Big|_{x=1}^{x=2} \\ &= \left[ \frac{1}{2}(2)^2 y^2 \right] - \left[ \frac{1}{2}(1)^2 y^2 \right] = \frac{3}{2}y^2\end{aligned}$$

Similarly, to evaluate  $\int_{-1}^1 xy^2 dy$ , you integrate with respect to  $y$ , holding  $x$  constant:

$$\begin{aligned}\int_{-1}^1 xy^2 dy &= x \left( \frac{1}{3}y^3 \right) \Big|_{y=-1}^{y=1} \\ &= \left[ x \left( \frac{1}{3}(1)^3 \right) \right] - \left[ x \left( \frac{1}{3}(-1)^3 \right) \right] = \frac{2}{3}x\end{aligned}$$

In general, partially integrating a function  $f(x, y)$  with respect to  $x$  results in a function of  $y$  alone, which can then be integrated as a function of a single variable,

thus producing what we call an **iterated integral**  $\int \left[ \int f(x, y) dx \right] dy$ . Similarly, the

iterated integral  $\int \left[ \int f(x, y) dy \right] dx$  is obtained by first integrating with respect to  $y$ ,

holding  $x$  constant, and then with respect to  $x$ . For instance,

$$\int_{-1}^1 \left( \int_1^2 xy^2 dx \right) dy = \int_{-1}^1 \frac{3}{2}y^2 dy = \frac{1}{2}y^3 \Big|_{y=-1}^{y=1} = 1$$

or

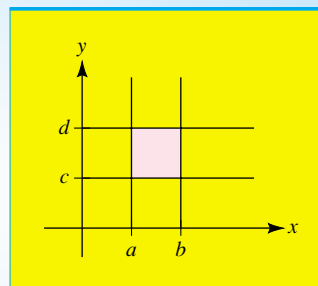
$$\int_1^2 \left( \int_{-1}^1 xy^2 dy \right) dx = \int_1^2 \frac{2}{3}x dx = \frac{1}{3}x^2 \Big|_{x=1}^{x=2} = 1$$

In our example, the two iterated integrals turned out to have the same value, and it can be shown that this will be true as long as the limits of integration are constants. We use this fact to define the double integral over a rectangular region as follows:

### The Double Integral over a Rectangular Region ■ The

double integral  $\iint_R f(x, y) dA$  over the rectangular region

$$R: a \leq x \leq b, c \leq y \leq d$$



is given by the common value of the two iterated integrals

$$\int_a^b \left[ \int_c^d f(x, y) dy \right] dx \quad \text{and} \quad \int_c^d \left[ \int_a^b f(x, y) dx \right] dy;$$

$$\text{that is, } \iint_R f(x, y) dA = \int_a^b \left[ \int_c^d f(x, y) dy \right] dx = \int_c^d \left[ \int_a^b f(x, y) dx \right] dy$$

Here is another example.

### EXAMPLE 5.1

Evaluate the double integral

$$\iint_R xe^{-y} dA$$

where  $R$  is the rectangular region  $-2 \leq x \leq 1$ ,  $0 \leq y \leq 5$  using:

- (a)  $x$  integration first  
 (b)  $y$  integration first

**Solution**

(a) With  $x$  integration first:

$$\begin{aligned} \iint_R xe^{-y} dA &= \int_0^5 \left( \int_{-2}^1 xe^{-y} dx \right) dy \\ &= \int_0^5 \frac{1}{2} x^2 e^{-y} \Big|_{x=-2}^{x=1} dy \\ &= \int_0^5 \frac{1}{2} e^{-y} [(1)^2 - (-2)^2] dy = \int_0^5 -\frac{3}{2} e^{-y} dy \\ &= -\frac{3}{2} (-e^{-y}) \Big|_{y=0}^{y=5} = \frac{3}{2} (e^{-5} - e^0) = \frac{3}{2} (e^{-5} - 1) \end{aligned}$$

(b) Integrating with respect to  $y$  first, you get

$$\begin{aligned} \iint_R xe^{-y} dA &= \int_{-2}^1 \left( \int_0^5 xe^{-y} dy \right) dx \\ &= \int_{-2}^1 x(-e^{-y}) \Big|_{y=0}^{y=5} dx = \int_{-2}^1 [-x(e^{-5} - e^0)] dx \\ &= \left[ -(e^{-5} - 1) \frac{1}{2} x^2 \right]_{x=-2}^{x=1} \\ &= -\frac{1}{2} (e^{-5} - 1) [(1)^2 - (-2)^2] = \frac{3}{2} (e^{-5} - 1) \end{aligned}$$

### CHOOSING THE ORDER OF INTEGRATION

In the examples you have seen so far, the order of integration makes little if any difference. Not only do the computations yield the same result, but the integrations are essentially of the same level of difficulty. However, as the following example illustrates, sometimes the order does matter.

### EXAMPLE 5.2

Evaluate the double integral

$$\iint_R xe^{xy} dA$$

where  $R$  is the rectangular region  $0 \leq x \leq 2$ ,  $0 \leq y \leq 1$ .

**Solution**

If you evaluate the integral in the order

$$\int_0^1 \left( \int_0^2 x e^{xy} dx \right) dy$$

it will be necessary to use integration by parts for the inner integration:

$$g(x) = e^{xy} \quad f(x) = x$$

$$G(x) = \frac{1}{y} e^{xy} \quad f'(x) = 1$$

$$\begin{aligned} \int_0^2 x e^{xy} dx &= \frac{x}{y} e^{xy} \Big|_{x=0}^{x=2} - \int_0^2 \frac{1}{y} e^{xy} dx \\ &= \left( \frac{x}{y} - \frac{1}{y^2} \right) e^{xy} \Big|_{x=0}^{x=2} = \left( \frac{2}{y} - \frac{1}{y^2} \right) e^{2y} - \left( \frac{-1}{y^2} \right) \end{aligned}$$

Then the outer integration becomes

$$\int_0^1 \left[ \left( \frac{2}{y} - \frac{1}{y^2} \right) e^{2y} + \frac{1}{y^2} \right] dy$$

Now what? Any ideas?

On the other hand, if you use  $y$  integration first, both computations are easy:

$$\begin{aligned} \int_0^2 \left( \int_0^1 x e^{xy} dy \right) dx &= \int_0^2 \frac{x e^{xy} \Big|_{y=0}^{y=1}}{x} dx \\ &= \int_0^2 [e^x - 1] dx = e^x - x \Big|_{x=0}^{x=2} \\ &= (e^2 - 2) - e^0 = e^2 - 3 \end{aligned}$$

## APPLICATIONS OF DOUBLE INTEGRALS

Recall from Section 1 of Chapter 6 that the definite integral  $\int_a^b f(x) dx$  of a non-negative function  $f(x)$  of one variable gives the area of the region under the graph of  $y = f(x)$  and above the interval  $a \leq x \leq b$ . A analogous argument for nonnegative functions of two variables yields the following formula for volume.

**Volume Formula** ■ If  $f(x, y) \geq 0$  for all points  $(x, y)$  in the rectangular region  $R$ , then the solid under the graph of  $f$  and above the region  $R$  has volume given by

$$V = \iint_R f(x, y) dA$$

**EXAMPLE 5.3**

Find the volume under the graph of  $f(x, y) = \frac{x}{y}$  and above the rectangular region  $R: 0 \leq x \leq 1, 1 \leq y \leq 2$ .

**Solution**

Since  $f(x, y) \geq 0$  for all  $(x, y)$  in  $R$ , the volume is given by the double integral

$$\begin{aligned} V &= \iint_R \frac{x}{y} dA = \int_1^2 \left( \int_0^1 \frac{x}{y} dx \right) dy \\ &= \int_1^2 \left[ \frac{1}{y} \left( \frac{1}{2} x^2 \right) \right]_{x=0}^{x=1} dy = \int_1^2 \frac{1}{2y} [(1)^2 - (0)^2] dy \\ &= \int_1^2 \frac{1}{2y} dy = \frac{1}{2} \ln y \Big|_{y=1}^{y=2} = \frac{1}{2} \ln 2 - \frac{1}{2} \ln 1 \\ &= \frac{1}{2} \ln 2 \approx 0.347 \text{ cubic units} \end{aligned}$$

**AVERAGE VALUE** In Section 3 of Chapter 6, you saw that the average value of a function  $f(x)$  over an interval  $a \leq x \leq b$  is given by the integral formula

$$\text{AV} = \frac{1}{b-a} \int_a^b f(x) dx$$

That is, to find the average value of a function of one variable over an interval, you integrate the function over the interval and divide by the length of the interval. The two-variable procedure is similar. In particular, to find the average value of a function of two variables  $f(x, y)$  over a rectangular region  $R$ , you integrate the function over  $R$  and divide by the area of  $R$ .

**Average Value Formula** ■ The average value of the function  $f(x, y)$  over the rectangular region  $R$  is given by the formula

$$\text{AV} = \frac{1}{\text{area of } R} \iint_R f(x, y) dA$$

### EXAMPLE 5.4

In a certain factory, output is given by the Cobb-Douglas production function

$$Q(K, L) = 50K^{3/5}L^{2/5}$$

where  $K$  is the capital investment in units of \$1,000 and  $L$  is the size of the labor force measured in worker-hours. Suppose that monthly capital investment varies between \$10,000 and \$12,000, while monthly use of labor varies between 2,800 and 3,200 worker-hours. Find the average monthly output for the factory.

#### Solution

It is reasonable to estimate the monthly output by the average value of  $Q(K, L)$  over the rectangular region  $R$ :  $10 \leq K \leq 12$ ,  $2,800 \leq L \leq 3,200$ . The region has area

$$\begin{aligned} A &= \text{area of } R = (12 - 10) \times (3,200 - 2,800) \\ &= 800 \end{aligned}$$

so the average output is

$$\begin{aligned} AV &= \frac{1}{800} \iint_R 50K^{3/5}L^{2/5} dA \\ &= \frac{1}{800} \int_{2,800}^{3,200} \left( \int_{10}^{12} 50K^{3/5}L^{2/5} dK \right) dL \\ &= \frac{1}{800} \int_{2,800}^{3,200} 50L^{2/5} \left( \frac{5}{8} K^{8/5} \right) \Big|_{K=10}^{K=12} dL \\ &= \frac{1}{800} (50) \left( \frac{5}{8} \right) \int_{2,800}^{3,200} L^{2/5} (12^{8/5} - 10^{8/5}) dL \\ &= \frac{1}{800} (50) \left( \frac{5}{8} \right) (12^{8/5} - 10^{8/5}) \left( \frac{5}{7} L^{7/5} \right) \Big|_{L=2,800}^{L=3,200} \\ &= \frac{1}{800} (50) \left( \frac{5}{8} \right) \left( \frac{5}{7} \right) (12^{8/5} - 10^{8/5}) [(3,200)^{7/5} - (2,800)^{7/5}] \\ &= 5,181.23 \end{aligned}$$

Thus, the average monthly output is approximately 5,181 units.

## JOINT PROBABILITY DENSITY FUNCTIONS

One of the most important applications of integration in the social, managerial, and life sciences is the computation of probabilities. The technique of integrating probability density functions to find probabilities was introduced in Chapter 6, Section 4. Recall that a probability density function for a random variable  $X$  is a nonnegative function  $f(x)$  such that the probability that  $X$  is between  $a$  and  $b$  is given by the formula

$$P(a \leq X \leq b) = \int_a^b f(x) \, dx$$

In situations involving two random variables  $X$  and  $Y$ , you compute probabilities by evaluating double integrals of a two-variable density function. In particular, you integrate a **joint probability density function**, which is a nonnegative function  $f(x, y)$  such that the probability that  $X$  is between  $a$  and  $b$  and  $Y$  is between  $c$  and  $d$  is given by the formula

$$\begin{aligned} P(a \leq X \leq b \text{ and } c \leq Y \leq d) &= \int_c^d \int_a^b f(x, y) \, dx \, dy \\ &= \int_a^b \int_c^d f(x, y) \, dy \, dx \end{aligned}$$

The techniques for constructing joint probability density functions from experimental data are beyond the scope of this book and are discussed in most probability and statistics texts. The use of double integrals to compute probabilities once the appropriate density functions are known is illustrated in the next example.

### EXAMPLE 5.5

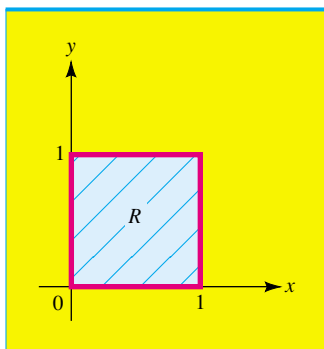
Smoke detectors manufactured by a certain firm contain two independent circuits, one manufactured at the firm's California plant and the other at the firm's plant in Ohio. Reliability studies suggest that if  $X$  measures the life span (in years) of a randomly selected circuit from the California plant and  $Y$  the life span (in years) of a randomly selected circuit from the Ohio plant, then the joint probability density function for  $X$  and  $Y$  is

$$f(x, y) = \begin{cases} e^{-x}e^{-y} & \text{if } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

If the smoke detector will operate as long as either of its circuits is operating, find the probability that a randomly selected smoke detector will fail within 1 year.

#### Solution

Since the smoke detector will operate as long as either of its circuits is operating, it will fail within 1 year if and only if *both* of its circuits fail within 1 year. The desired probability is thus the probability that both  $0 \leq X \leq 1$  and  $0 \leq Y \leq 1$ . The points



**FIGURE 7.25** Square consisting of all points  $(x, y)$  for which  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ .

$(x, y)$  for which both these inequalities hold form the square  $R$  shown in Figure 7.25. The corresponding probability is the double integral of the density function  $f$  over this region  $R$ . That is,

$$\begin{aligned}
 P(0 \leq X \leq 1 \text{ and } 0 \leq Y \leq 1) &= \int_0^1 \int_0^1 e^{-x}e^{-y} dx dy \\
 &= \int_0^1 \left( -e^{-x}e^{-y} \Big|_{x=0}^{x=1} \right) dy \\
 &= \int_0^1 -(e^{-1} - 1)e^{-y} dy \\
 &= (e^{-1} - 1)e^{-y} \Big|_{y=0}^{y=1} = (e^{-1} - 1)^2 \\
 &= 0.3996
 \end{aligned}$$

Thus, it is approximately 40% likely that a randomly selected smoke detector will fail within 1 year.

## P . R . O . B . L . E . M . S 7.5

Evaluate the following double integrals.

- |  |  |
|--|--|
| 1. $\int_0^1 \int_1^2 x^2y dx dy$                | 2. $\int_1^2 \int_0^1 x^2y dy dx$                                      |
| 3. $\int_0^{\ln 2} \int_{-1}^0 2xe^y dx dy$      | 4. $\int_2^3 \int_{-1}^1 (x + 2y) dy dx$                               |
| 5. $\int_1^3 \int_0^1 \frac{2xy}{x^2 + 1} dx dy$ | 6. $\int_0^1 \int_0^1 x^2e^{xy} dy dx$                                 |
| 7. $\int_0^4 \int_{-1}^1 x^2y dy dx$             | 8. $\int_0^1 \int_1^5 y\sqrt{1 - y^2} dx dy$                           |
| 9. $\int_2^3 \int_1^2 \frac{x + y}{xy} dy dx$    | 10. $\int_1^2 \int_2^3 \left( \frac{y}{x} + \frac{x}{y} \right) dy dx$ |

In Problems 11 through 16, find the volume of the solid bounded above by the graph of the function  $f(x, y)$  and below by the given rectangular region  $R$ .

11.  $f(x, y) = 6 - 2x - 2y$ ;  $R$  has vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 2)$ ,  $(1, 2)$   
 12.  $f(x, y) = 9 - x^2 - y^2$ ;  $R$  has vertices  $(-1, -2)$ ,  $(1, -2)$ ,  $(-1, 1)$ ,  $(1, 1)$



$$13. f(x, y) = \frac{1}{xy}; R: 1 \leq x \leq 2, 1 \leq y \leq 3$$

$$14. f(x, y) = e^{x+y}; R: 0 \leq x \leq 1, 0 \leq y \leq \ln 2$$

$$15. f(x, y) = xe^{-y}; R: 0 \leq x \leq 1, 0 \leq y \leq 2$$

$$16. f(x, y) = (1 - x)(4 - y); R: 0 \leq x \leq 1, 0 \leq y \leq 4$$

In Problems 17 through 20, find the average value of the function  $f(x, y)$  over the given rectangular region  $R$ .

$$17. f(x, y) = xy(x - 2y); R \text{ has vertices } (-2, 2), (3, 2), (-2, -1), (3, -1)$$

$$18. f(x, y) = \frac{y}{x} + \frac{x}{y}; R \text{ has vertices } (1, 1), (4, 1), (1, 3), (4, 3)$$

$$19. f(x, y) = xe^{y^2}; R: 0 \leq x \leq 1, 0 \leq y \leq 2$$

$$20. f(x, y) = \frac{\ln x}{xy}; R: 1 \leq x \leq 2, 2 \leq y \leq 3$$

In Problems 21 and 22, evaluate the given double integral over the specified rectangular region  $R$ . Choose the order of integration carefully.

$$21. \iint_R \frac{\ln xy}{xy^2} dA \text{ over } R: 1 \leq x \leq 3, 2 \leq y \leq 5$$

$$22. \iint_R ye^{xy} dA \text{ over } R: -1 \leq x \leq 1, 1 \leq y \leq 2$$

23. Suppose the joint probability density function for the nonnegative random variables  $X$  and  $Y$  is

$$f(x, y) = \begin{cases} 2e^{-2x}e^{-y} & \text{if } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that  $0 \leq X \leq 1$  and  $1 \leq Y \leq 2$ .

24. Suppose the joint probability density function for the nonnegative random variables  $X$  and  $Y$  is

$$f(x, y) = \begin{cases} \frac{1}{6}e^{-x/2}e^{-y/3} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that  $1 \leq X \leq 2$  and  $0 \leq Y \leq 2$ .

25. Evaluate the double integral  $\iint_R x^2e^{x^2y} dA$ , where  $R$  is the rectangular region  $0 \leq x \leq 1, 0 \leq y \leq 1$ . (Be careful how you choose the order of integration.)

26. Evaluate the double integral  $\iint_R \frac{dA}{xy \ln x}$ , where  $R$  is the rectangular region  $2 \leq x \leq 3$ ,  $1 \leq y \leq 2$ .

**PRODUCTION** 27. At a certain factory, output  $Q$  is related to inputs  $x$  and  $y$  by the expression  $Q(x, y) = 2x^3 + 3x^2y + y^3$ . If  $0 \leq x \leq 5$  and  $0 \leq y \leq 7$ , what is the average output of the factory?

**PRODUCTION** 28. A bicycle dealer has found that if 10-speed bicycles are sold for  $x$  dollars apiece and the price of gasoline is  $y$  cents per gallon, then approximately

$$Q(x, y) = 200 - 24\sqrt{x} + 4(0.1y + 5)^{3/2}$$

bicycles will be sold each month. If the price of bicycles varies between \$257 and \$312 during a typical month, and the price of gasoline varies between \$1.03 and \$1.21, approximately how many bicycles will be sold each month?

**AVERAGE PROFIT** 29. A manufacturer estimates that when  $x$  units of a particular commodity are sold domestically and  $y$  units are sold to foreign markets, the profit is given by

$$P(x, y) = (x - 30)(70 + 5x - 4y) + (y - 40)(80 - 6x + 7y)$$

hundred dollars. If monthly domestic sales vary between 100 and 125 units and foreign sales, between 70 and 89 units, what is the average monthly profit?

**AVERAGE RESPONSE TO STIMULI** 30. In a psychological experiment,  $x$  units of stimulus  $A$  and  $y$  units of stimulus  $B$  are applied to a subject, whose performance on a certain task is then measured by the function

$$P(x, y) = 10 + xye^{1-x^2-y^2}$$

Suppose  $x$  varies between 0 and 1 while  $y$  varies between 0 and 3. What is the subject's average response to the stimuli?

**AVERAGE ELEVATION** 31. A map of a small regional park is a rectangular grid, bounded by the lines  $x = 0$ ,  $x = 4$ ,  $y = 0$ , and  $y = 3$ , where units are in miles. It is found that the elevation above sea level at each point  $(x, y)$  in the park is given by

$$E(x, y) = 90(2x + y^2) \text{ feet}$$

Find the average elevation in the park. (Remember, 1 mi = 5,280 feet.)

**AVERAGE SURFACE AREA OF THE HUMAN BODY** 32. Recall from Problem 35, Section 1, that the surface area of a person's body may be estimated by the empirical formula

$$S(W, H) = 0.0072 W^{0.425} H^{0.725}$$

where  $W$  is the person's weight in kilograms,  $H$  is the person's height in centimeters, and the surface area  $S$  is measured in square meters.

(a) A child weighs 3.2 kg and is 38 cm tall at birth, and 5 years later weighs 14 kg and is 115 cm tall. What is the average surface area of the child's body during this period?

(b) Suppose the child in part (a) has a stable adult weight of 80 kg and height of 180 cm. Find the average lifetime surface area of this person's body.

- HEALTH CARE** 33. Suppose the random variables  $X$  and  $Y$  measure the length of time (in days) that a patient stays in the hospital after abdominal and orthopedic surgery, respectively. On Monday, the patient in bed 107A undergoes an emergency appendectomy while her roommate in bed 107B undergoes (orthopedic) surgery for the repair of torn knee cartilage. If the joint probability density function for  $X$  and  $Y$  is

$$f(x, y) = \begin{cases} \frac{1}{12}e^{-x/4}e^{-y/3} & \text{if } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

what is the probability that both patients will be discharged from the hospital within 3 days?

- WARRANTY PROTECTION** 34. A certain appliance consisting of two independent electronic components will be usable as long as at least one of its components is still operating. The appliance carries a warranty from the manufacturer guaranteeing replacement if the appliance becomes unusable within 1 year of the date of purchase. Let the random variable  $X$  measure the life span (in years) of the first component and  $Y$  measure the life span (also in years) of the second component, and suppose that the joint probability density function for  $X$  and  $Y$  is

$$f(x, y) = \begin{cases} \frac{1}{4}e^{-x/2}e^{-y/2} & \text{if } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

You purchase one of these appliances, selected at random from the manufacturer's stock. Find the probability that the warranty will expire before your appliance becomes unusable. [*Hint:* You want the probability that the appliance *does not* fail during the first year. How is this related to the probability that the appliance *does* fail during this period?]

- PROPERTY VALUE** 35. Suppose a particular congressional district is laid out in a rectangular grid with a city at the origin. It is estimated that the property value at the point  $(x, y)$  in the grid is given by



$$V(x, y) = 90e^{-x^2-y}$$

thousand dollars, where  $-10 \leq x \leq 10$  and  $-5 \leq y \leq 5$ . Find the average property value for the district. [*Hint:* You will not be able to find an antiderivative for  $e^{-x^2}$ . Estimate the average value integral using the numeric integration feature of your calculator.]

- EXPOSURE TO DISEASE** 36. The likelihood that a person with a contagious disease will infect others in a social situation may be assumed to be a function  $f(s)$  of the distance  $s$  between individuals. Suppose contagious individuals are uniformly distributed through a rectangular region  $R$  in the  $xy$  plane. Then the likelihood of infection for someone at the origin  $(0, 0)$  is proportional to the exposure index  $E$ , given by the double integral



$$E = \iint_R f(s) dA$$

where  $s = \sqrt{x^2 + y^2}$  is the distance between  $(0, 0)$  and  $(x, y)$ ,

- (a) Find  $E$  for the case where  $R$  is the square region  $-2 \leq x \leq 2$ ,  $-2 \leq y \leq 2$ , and  $f(s) = 1 - \frac{1}{9}s^2$ .
- (b) Find  $E$  for the case where  $R$  is the region  $-1 \leq x \leq 1$ ,  $-3 \leq y \leq 2$ , and  $f(s) = se^{-5}$ .
- (c) Find  $E$  for the case where  $R$  is the region in part (b) but  $f(s) = e^{-0.15}$ . [Hint: Use the numeric integration feature of your calculator.]



In Problems 37 through 39, use double integration to find the required quantity. In some cases, you may need to use the numeric integration feature of your calculator.

37. Find the volume of the solid bounded above by the graph of  $f(x, y) = x^2e^{-xy}$  and below by the rectangular region  $R: 0 \leq x \leq 2$ ,  $0 \leq y \leq 3$ .
38. Find the average value of the function  $f(x, y) = xy \ln\left(\frac{y}{x}\right)$  over the rectangular region bounded by the lines  $x = 1$ ,  $x = 2$ ,  $y = 1$ , and  $y = 3$ .
39. Suppose the joint probability density function for the random variables  $X$  and  $Y$  is

$$f(x, y) = \begin{cases} xe^{-x-y} & \text{if } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that  $(X, Y)$  lies in the rectangular region  $R$  with vertices  $(0,0)$ ,  $(1,0)$ ,  $(0,2)$ , and  $(1,2)$ .

## CHAPTER SUMMARY AND REVIEW PROBLEMS

### IMPORTANT TERMS, SYMBOLS, AND FORMULAS

Function of two variables:  $z = f(x, y)$

Surface

Level curve:  $f(x, y) = C$

Constant-production curve (isoquant)

Indifference curve

Partial derivatives of  $z = f(x, y)$ :

$$f_x = \frac{\partial z}{\partial x} \quad f_y = \frac{\partial z}{\partial y}$$

Second-order partial derivatives: