Response Optimization in Differential Equations

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1 introduction

Consider

$$my'' + cy' + ky = F \cos \omega t,$$

subject to $y(0) = a, y'(0) = b$, for positive constants m, c, k, ω. Here $F \cos \omega t$ is the input to a system and the solution y is the output. We want to adjust ω so that the output is the strongest possible. We do this in three steps:
Part A: solving the equation and finding the optimal value of ω,
Part B: writing a visualizing program to show the solution,
Part C: verifying that visualization results agree with our theoretical solution.

2 Part A, Analysis of DE

In this section we solve the differential equation $my'' + cy' + ky = F \cos \omega t$, subject to $y(0) = a, y'(0) = b$, and analyze the resulting components. The values of $m,\ c,\ k$ are positive and for simplicity F is also assumed positive.

1. Find the solution as a sum of two parts, a transient or decaying part $T(t)$ and a steady state or permanent part $S(t)$. Explain how you detect the two parts. Find the general form of $T(t)$. Find the steady state response $S(t) = A \cos \omega t + B \sin \omega t$, and give A and B in terms of $\omega,\ m,\ c,\ k,\ F$. Simplify. Apply the initial conditions.

   Note that (1) the initial conditions are applied at the end, to $T + S$; and (2) T can have one of three formulas, depending on the values of $m,\ c,\ k$. So at this stage you have three formulas, each applying for a certain range of parameters.

2. Write $S(t)$ as $R \cos(\omega t - d)$, with $R > 0$. Find R and d in terms of ω, m, c, k, F. Simplify.

3. Define $N = Rk/F$, $\omega_0 = \sqrt{k/m}$, $z = \omega/\omega_0$, and $g = c^2/mk$.

   (a) Write N in terms of z and g.

(b) Plot N as a function of z for several values of g including $g = 2, g = 0.5, g = 0.1, g = 0.01$. Plots must be on a single graph.

(c) Express d in terms of z and g. In which quadrant is d? Explain.

(d) Plot d as a function of z for several values of g including $g = 2, g = 0.5, g = 0.1, g = 0.01$. Plots must be on a single graph.

4. Consider R as a function of ω.

   (a) Given $m,\ c,\ k$ find the value of ω which makes R the maximum. We will refer to this value as ω_{max}.

   (b) For what values of $m,\ c,\ k$ does ω_{max} exist? Explain.

   (c) Find $R,\ d,\ N,\ z$ for $\omega = \omega_{max}$. Simplify.

   (d) Express d, N, and z for $\omega = \omega_{max}$ in terms of g.

3 Part B, visualization

Write a Desmos program to display the general solution, $y(t)$, with sliders for ω, m, c, k, a, b, F. You may want to break your formulas into multiple steps so that it remains visible in the formula column of Desmos.
Use the hints provided on course website to create piece-wise defined function in Desmos to write the general solution y.

4 Part C, verification

We have taken many algebra steps in Part A, and we have written a program in Part B. How do we know we have not made any mistakes and that our approach is meaningful. One remedy is to show, with graphical experimentation, that as we change ω in Desmos the maximum value of R does show up at ω_{max} and it agrees with the formulas we found. This is what you will show in your presentation. In your report include several examples of this event for different values of the parameters.