# 2 Applications to Business and Economics

In Section 6.1, you saw that area can be expressed as the limit of a sum, then evaluated with antidifferentiation by applying the fundamental theorem of calculus. This procedure, called definite integration, was introduced through area because area is easy to visualize, but there are many other applications in which the integration process plays an important role. In this section, we focus on applications of integration to business and economics. Additional applications to a variety of other subjects are examined in Section 6.3.

## Applying the Definite Integral

Intuitively, the definite integral can be thought of as a process that "adds up" an infinite number of small pieces of a given quantity to obtain the "total" quantity. Here is a step by step procedure for using this procedure in practical situations.

**A Procedure for Using Definite Integration in Applications** ■ To use definite integration to model the "totality" of a quantity $f(x)$ over an interval $a \leq x \leq b$ where it is continuous, proceed as follows:

**Step 1** Divide the interval $a \leq x \leq b$ into $n$ equal subintervals, each of length $\Delta x = \dfrac{b - a}{n}$. For $j = 1, 2, \ldots, n$, choose a number $x_j$ from the $j$th subinterval.

**Step 2** Approximate the contribution to the total quantity that comes from the $j$th subinterval by $f(x_j)\,\Delta x$. Then add the individual contributions to estimate the total quantity by the sum

$$[f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

**Step 3** Take the limit as $n \to +\infty$ to pass from the approximation to the exact value of the quantity.

**Step 4** Use the fact that

$$\lim_{n \to +\infty} [f(x_1) + \cdots + f(x_n)]\Delta x = \int_a^b f(x)\,dx$$

to transform the limit of a sum in step 3 to a definite integral. Then evaluate the definite integral by the fundamental theorem of calculus to obtain the required total quantity.

## The Amount of an Income Stream

In our first application, we use definite integration to examine a stream of income transferred continuously into an account in which it earns interest over a specified time period (the **term**). The **amount** (or **future value**) **of an income stream** is the total amount (money transferred into the account plus interest) that is accumulated in this way during the specified term.

The calculation of the amount of an income stream is illustrated in the following example. The strategy is to approximate the continuous income stream by a sequence of discrete deposits called an **annuity.** The amount of the approximating annuity is a certain sum whose limit (a definite integral) is the amount of the income stream.

## EXAMPLE 2.1

Money is transferred continuously into an account at the constant rate of \$1,200 per year. The account earns interest at the annual rate of 8% compounded continuously. How much will be in the account at the end of 2 years?

### Solution

Recall from Chapter 4 that $P$ dollars invested at 8% compounded continuously will be worth $Pe^{0.08t}$ dollars $t$ years later.

To approximate the future value of the income stream, divide the 2-year time interval $0 \le t \le 2$ into $n$ equal subintervals of length $\Delta_n t$ years and let $t_j$ denote the beginning of the $j$th subinterval. Then, during the $j$th subinterval (of length $\Delta_n t$ years),

$$\text{Money deposited} = (\text{dollars per year})(\text{number of years}) = 1{,}200\,\Delta_n t$$

If all of this money were deposited at the beginning of the subinterval (at time $t_j$), it would remain in the account for $2 - t_j$ years and therefore would grow to $(1{,}200\,\Delta_n t)e^{0.08(2-t_j)}$ dollars. Thus,

$$\begin{matrix}\text{Future value of money deposited} \\ \text{during } j\text{th subinterval}\end{matrix} \simeq 1{,}200e^{0.08(2-t_j)}\,\Delta_n t$$

The situation is illustrated in Figure 6.8.

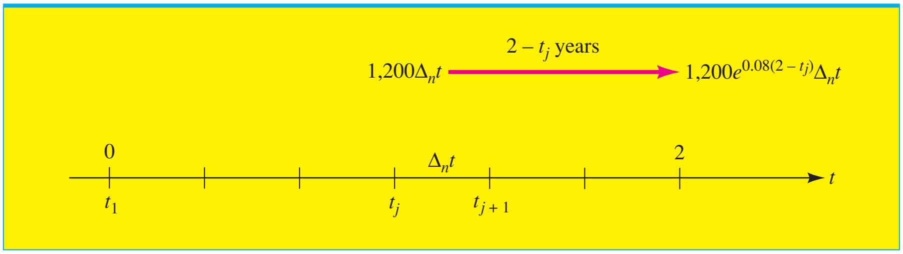



**FIGURE 6.8** The (approximate) future value of the money deposited during the $j$th subinterval.

The future value of the entire income stream is the sum of the future values of the money deposited during each of the $n$ subintervals. Hence,

$$\text{Future value of income stream} \simeq \sum_{j=1}^{n} 1{,}200e^{0.08(2-t_j)}\,\Delta_n t$$

(Note that this is only an approximation because it is based on the assumption that all $1{,}200\,\Delta_n t$ dollars are deposited at time $t_j$ rather than continuously throughout the $j$th subinterval.)

As $n$ increases without bound, the length of each subinterval approaches zero and the approximation approaches the true future value of the income stream. Hence,

$$\begin{aligned}\text{Future value of income stream} &= \lim_{n \to +\infty} \sum_{j=1}^{n} 1{,}200e^{0.08(2-t_j)}\Delta_n t \\ &= \int_0^2 1{,}200e^{0.08(2-t)}\,dt = 1{,}200e^{0.16}\int_0^2 e^{-0.08t}\,dt \\ &= -\frac{1{,}200}{0.08}e^{0.16}(e^{-0.08t})\Big|_0^2 = 15{,}000e^{0.16}(e^{-0.16}-1) \\ &= -15{,}000 + 15{,}000e^{0.16} \simeq \$2{,}602.66\end{aligned}$$

The **present value** of an income stream generated continuously at a certain rate over a specified period of time is the amount of money that must be deposited today at the prevailing interest rate to generate the same income stream over the same term. The calculation of the present value of a continuous income stream is illustrated in Example 2.2. As with future value, the strategy is to approximate the continuous income stream by a sequence of discrete payments; that is, by an annuity. The present value of the approximating annuity is a certain sum, and by taking the limit of this sum, we obtain a definite integral that represents the present value of the income stream.

## EXAMPLE 2.2

The management of a national chain of ice cream parlors is selling a 5-year franchise to operate its newest outlet in Madison, Wisconsin. Past experience in similar localities suggests that $t$ years from now the franchise will be generating profit at the rate of $f(t) = 14{,}000 + 490t$ dollars per year. If the prevailing annual interest rate remains fixed during the next 5 years at 7% compounded continuously, what is the present value of the franchise?

### Solution

Recall from Chapter 4 that if the prevailing annual interest rate is 7% compounded continuously, the present value of $B$ dollars payable $t$ years from now is $Be^{-0.07t}$.

To approximate the present value of the franchise, divide the 5-year time interval $0 \le t \le 5$ into $n$ equal subintervals of length $\Delta_n t$ years and let $t_j$ denote the beginning of the $j$th subinterval. Then,

$$\text{Profit from the } j\text{th interval} = (\text{dollars per year})(\text{number of years}) \simeq f(t_j)\,\Delta_n t$$

Hence

$$\text{Present value of the profit from the } j\text{th subinterval} \simeq f(t_j)e^{-0.07t_j}\,\Delta_n t$$

and $$\text{Present value of franchise} \simeq \sum_{j=1}^{n} f(t_j)e^{-0.07t_j}\,\Delta_n t$$

The situation is illustrated in Figure 6.9.

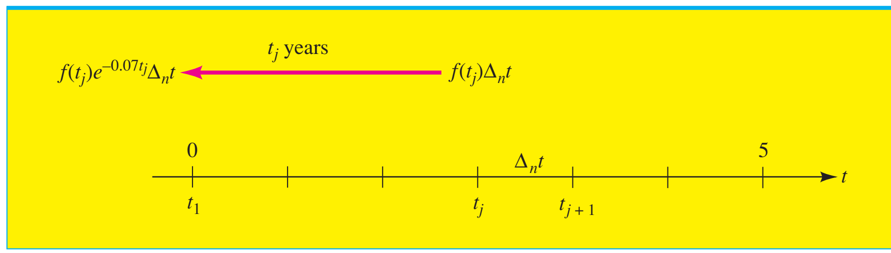



**FIGURE 6.9** The (approximate) present value of the profit generated during the $j$th subinterval.

The approximation approaches the true present value as $n$ increases without bound. Thus,

$$
\begin{aligned}
\text{Present value of franchise} &= \lim_{n \to +\infty} \sum_{j=1}^{n} f(t_j)e^{-0.07t_j}\,\Delta_n t \\
&= \int_0^5 f(t)e^{-0.07t}\,dt \\
&= \int_0^5 (14{,}000 + 490t)e^{-0.07t}\,dt \\
&= -\frac{1}{0.07}(14{,}000 + 490t)e^{-0.07t}\Big|_0^5 + 7{,}000\int_0^5 e^{-0.07t}\,dt \qquad \text{(integration by parts)} \\
&= [-1{,}000(200 + 7t)e^{-0.07t} - 100{,}000e^{-0.07t}]\Big|_0^5 \\
&= -1{,}000(300 + 7t)e^{-0.07t}\Big|_0^5 \\
&= -1{,}000(335e^{-0.35} - 300) \simeq \$63{,}929.49
\end{aligned}
$$

## Net Change

In many practical applications, we are given the rate of change $Q'(x)$ of a quantity $Q(x)$ and required to compute the **net change** $NC = Q(b) - Q(a)$ in $Q(x)$ as $x$ varies from $x = a$ to $x = b$. But since $Q(x)$ is an antiderivative of $Q'(x)$, the fundamental theorem of calculus tells us that the net change is given by the definite integral

$$NC = Q(b) - Q(a) = \int_a^b Q'(x)\,dx$$

Here are several examples illustrating the computation of net change by definite integration.

## EXAMPLE 2.3

At a certain factory, the marginal cost is $3(q-4)^2$ dollars per unit when the level of production is $q$ units. By how much will the total manufacturing cost increase if the level of production is raised from 6 units to 10 units?

### Solution

Let $C(q)$ denote the total cost of producing $q$ units. Then the marginal cost is the derivative $\frac{dC}{dq} = 3(q-4)^2$, and the increase in cost if production is raised from 6 units to 10 units is given by the definite integral

$$C(10) - C(6) = \int_6^{10} 3(q-4)^2\,dq = (q-4)^3 \Big|_6^{10}$$
$$= (10-4)^3 - (6-4)^3 = 216 - 8 = \$208$$

### NET EXCESS PROFIT

Suppose that $t$ years from now, two investment plans will be generating profit $P_1(t)$ and $P_2(t)$, respectively, and that the respective rates of profitability, $P_1'(t)$ and $P_2'(t)$, satisfy $P_2'(t) \geq P_1'(t)$ for the first $N$ years $(0 \leq t \leq N)$. Then

$$E(t) = P_2(t) - P_1(t)$$

represents the **excess profit** of plan 2 over plan 1 at time $t$ and the **net excess profit** $\text{NE} = E(N) - E(0)$ over the time period $0 \leq t \leq N$ is given by the definite integral

$$\text{NE} = E(N) - E(0) = \int_0^N E'(t)\,dt$$
$$= \int_0^N [P_2'(t) - P_1'(t)]\,dt$$

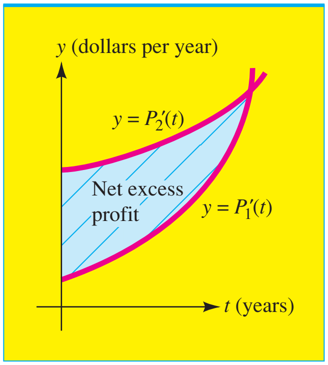



**FIGURE 6.10** Net excess profit as the area between two curves.

which can be interpreted geometrically as the area between the curves $y = P_1'(t)$ and $y = P_2'(t)$ (see Figure 6.10). Here is an example.

## EXAMPLE 2.4

Suppose that $t$ years from now, one investment will be generating profit at the rate of $P_1'(t) = 50 + t^2$ hundred dollars per year, while a second investment will be generating profit at the rate of $P_2'(t) = 200 + 5t$ hundred dollars per year.

**(a)** For how many years does the rate of profitability of the second investment exceed that of the first?

**(b)** Compute the net excess profit for the time period determined in part (a). Interpret the net excess profit as an area.

**Solution**

**(a)** The rate of profitability of the second investment exceeds that of the first until

$$P_1'(t) = P_2'(t)$$
$$50 + t^2 = 200 + 5t$$
$$t^2 - 5t - 150 = 0$$
$$(t - 15)(t + 10) = 0$$
$$t = 15 \text{ years} \quad (\text{reject } t = -10)$$

**(b)** The net excess profit for the time period $0 \leq t \leq 15$ is given by the definite integral

$$\text{NE} = \int_0^{15} [P_2'(t) - P_1'(t)]\,dt = \int_0^{15} [(200 + 5t) - (50 + t^2)]\,dt$$
$$= \int_0^{15} (150 + 5t - t^2)\,dt$$
$$= \left(150t + \frac{5}{2}t^2 - \frac{1}{3}t^3\right)\Bigg|_0^{15} = 1{,}687.50 \text{ hundred dollars}$$

that is, \$168,750.

The rate of profit curves for the two investments are shown in Figure 6.11. The net excess profit is the area of the (shaded) region between the curves.

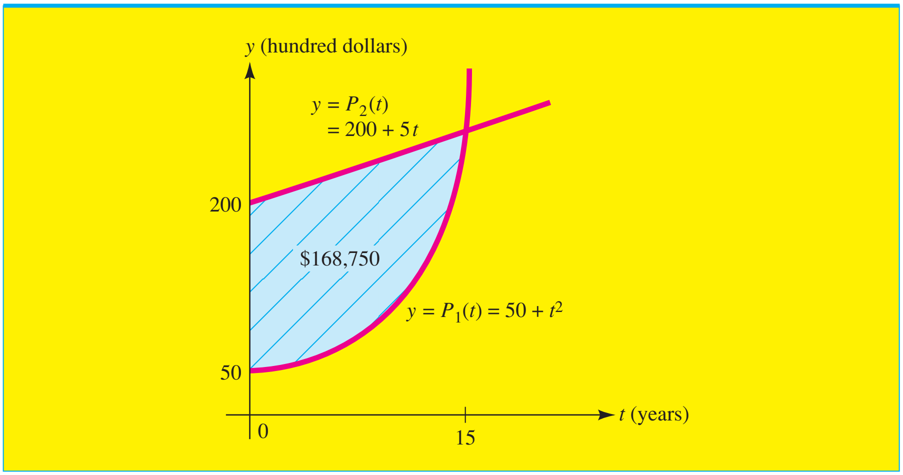



**FIGURE 6.11** Net excess profit for Example 2.4.

## Net Earnings from Industrial Equipment

The net earnings generated by an industrial machine over a period of time is the difference between the total revenue generated by the machine and the total cost of operating and servicing the machine. The following example shows how net earnings can be computed by definite integration.

**Explore!**

Refer to Example 2.5. Suppose a new cost rate function $C'_{\text{new}}(t) = 2{,}000 + 6t^2$ is in place. Compute the years of profitability and net earnings under this new cost function, compared with the original cost rate function. Use the window [0, 20]5 by [0, 8,000]1,000.

### EXAMPLE 2.5

Suppose that when it is $t$ years old, a particular industrial machine generates revenue at the rate $R'(t) = 5{,}000 - 20t^2$ dollars per year and that operating and servicing costs related to the machine accumulate at the rate $C'(t) = 2{,}000 + 10t^2$ dollars per year.

**(a)** How many years pass before the profitability of the machine begins to decline?

**(b)** Compute the net earnings generated by the machine over the time period determined in part (a).

#### Solution

**(a)** The profit associated with the machine after $t$ years of operation is $P(t) = R(t) - C(t)$ and the rate of profitability is

$$\begin{aligned} P'(t) = R'(t) - C'(t) &= (5{,}000 - 20t^2) - (2{,}000 + 10t^2) \\ &= 3{,}000 - 30t^2 \end{aligned}$$

The profitability begins to decline when

$$\begin{aligned} P'(t) &= 0 \\ 3{,}000 - 30t^2 &= 0 \\ t^2 &= 100 \\ t &= 10 \text{ years} \end{aligned}$$

**(b)** The net earnings NE over the time period $0 \le t \le 10$ is given by the difference $\text{NE} = P(10) - P(0)$, which can be computed by the integral

$$\begin{aligned} \text{NE} = P(10) - P(0) &= \int_0^{10} P'(t)\,dt \\ &= \int_0^{10} (3{,}000 - 30t^2)\,dt \\ &= (3{,}000t - 10t^3)\Big|_0^{10} = \$20{,}000 \end{aligned}$$

The rate of revenue and rate of cost curves are sketched in Figure 6.12. The net earnings is the area of the (shaded) region between the curves.

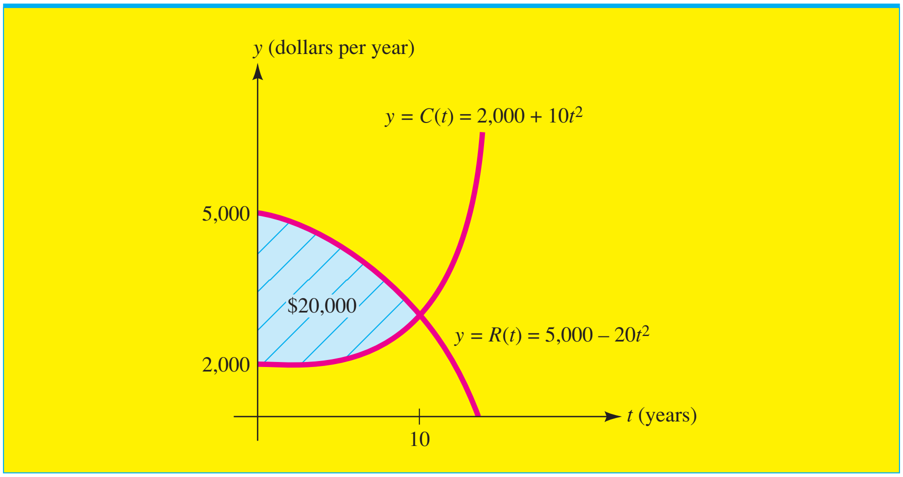



**FIGURE 6.12** The net earnings from an industrial machine.

## The Consumers' Demand Curve and Willingness to Spend

In studying consumer behavior, economists often assume that the price a consumer or group of consumers is willing to pay to buy an additional unit of a commodity is a function of the number of units of the commodity that the consumer or group has already bought.

For example, a young couple on a limited budget might be willing to spend up to \$500 to own one television set. For the convenience of having two sets (to eliminate the conflict between *Monday Night Football* and the network news, for instance), the couple might be willing to spend an additional \$300 to buy a second set. Since there would be very little use for more than two sets, the couple might be willing to spend no more than \$100 for a third set.

A function $p = D(q)$ giving the price per unit that consumers are willing to pay to get the $q$th unit of a commodity is known in economics as the **consumers' demand function** for the commodity. As illustrated in Figure 6.13, the consumers' demand function is usually a decreasing function of $q$. That is, the price that consumers are willing to pay to get one additional unit usually decreases as the number of units already bought increases.

The consumers' demand function $p = D(q)$ can also be thought of as the rate of change with respect to $q$ of the total amount $A(q)$ that consumers are willing to spend for $q$ units; that is,

$$\frac{dA}{dq} = D(q)$$

Integrating, you find that the total amount that consumers are willing to pay for $q_0$ units of the commodity is given by

$$A(q_0) - A(0) = \int_0^{q_0} \frac{dA}{dq}\,dq = \int_0^{q_0} D(q)\,dq$$

In this context, economists call $A(q)$ the **total willingness to spend** and $D(q) = A'(q)$ the **marginal willingness to spend.** In geometric terms, the total willingness to spend for $q_0$ units is the area under the demand curve $p = D(q)$ between $q = 0$ and $q = q_0$ (see Figure 6.13).

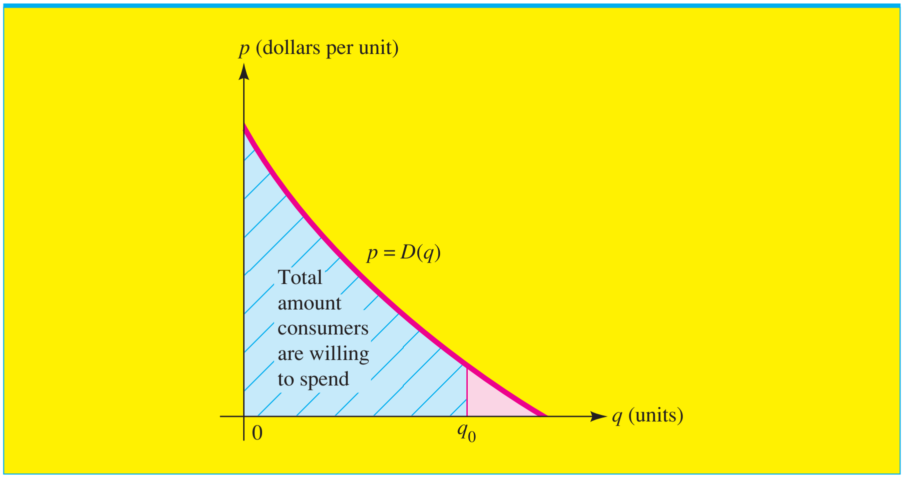



**FIGURE 6.13** The amount consumers are willing to spend is the area under the demand curve.

**Explore!**

Refer to Example 2.6. If $D(q)$ is changed to $D_1(q) = 4(23 - q^2)$, will there be an increase or decrease in the total amount of money consumers are willing to spend to get 3 units of the commodity. Use [0, 5]1 by [0, 150]10 as the size of your viewing window. Graph $D(q)$ and $D_1(q)$ (in bold) to assist your conjecture.

## EXAMPLE 2.6

Suppose that the consumers' demand function for a certain commodity is $D(q) = 4(25 - q^2)$ dollars per unit.

**(a)** Find the total amount of money consumers are willing to spend to get 3 units of the commodity.

**(b)** Sketch the demand curve and interpret the answer to part (a) as an area.

### Solution

**(a)** Since the demand function $D(q) = 4(25 - q^2)$, measured in dollars per unit, is the rate of change with respect to $q$ of consumers' willingness to spend, the total amount that consumers are willing to spend to get 3 units of the commodity is given by the definite integral

$$\int_0^3 D(q)\,dq = 4\int_0^3 (25 - q^2)\,dq = 4\left(25q - \frac{1}{3}q^3\right)\Bigg|_0^3 = \$264$$

**(b)** The consumers' demand curve is sketched in Figure 6.14. In geometric terms, the total amount, \$264, that consumers are willing to spend to get 3 units of the commodity is the area under the demand curve from $q = 0$ to $q = 3$.

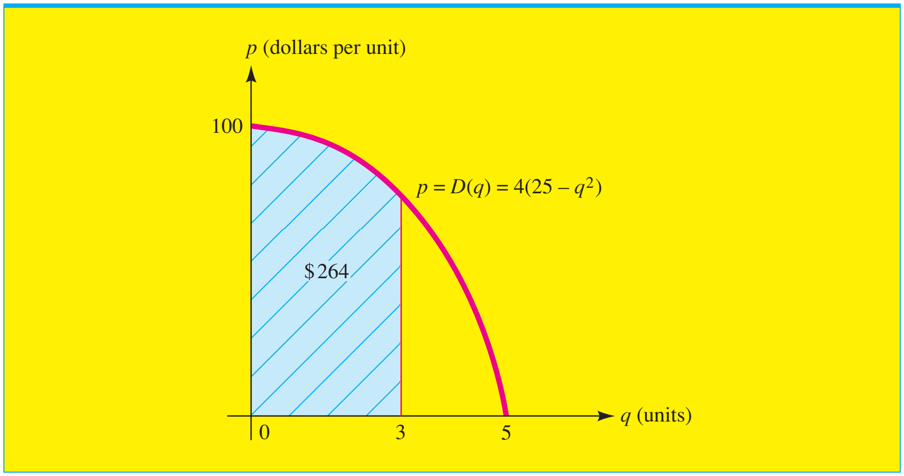



**FIGURE 6.14** Consumers' willingness to spend for 3 units when demand is given by $p = 4(25 - q^2)$.

## Consumers' and Producer's Surplus

In a competitive economy, the total amount that consumers actually spend on a commodity is usually less than the total amount they would have been willing to spend. The difference between the two amounts can be thought of as a savings realized by consumers and is known in economics as the **consumers' surplus.** That is,

$$\begin{bmatrix}\text{Consumers'}\\ \text{surplus}\end{bmatrix} = \begin{pmatrix}\text{total amount consumers}\\ \text{would be willing to spend}\end{pmatrix} - \begin{pmatrix}\text{actual consumer}\\ \text{expenditure}\end{pmatrix}$$

Market conditions determine the price per unit at which a commodity is sold. Once the price, say $p_0$, is known, the demand equation $p = D(q)$ determines the number of units $q_0$ that consumers will buy. The actual consumer expenditure for $q_0$ units of the commodity at the price of $p_0$ dollars per unit is $p_0q_0$ dollars. The consumers' surplus is calculated by subtracting this amount from the total amount consumers would have been willing to spend to get $q_0$ units of the commodity.

To get a better feel for the concept of consumers' surplus, consider once again the example of the couple that was willing to spend \$500 for their first television set, \$300 for a second set, and \$100 for a third set. Suppose the market price for television sets is \$300 per set. Then the couple would buy only two sets and would spend a total of $2 \times \$300 = \$600$. This is less than the $\$500 + \$300 = \$800$ that the couple would have been willing to spend to get the two sets. The savings of $\$800 - \$600 = \$200$ is the couple's consumers' surplus.

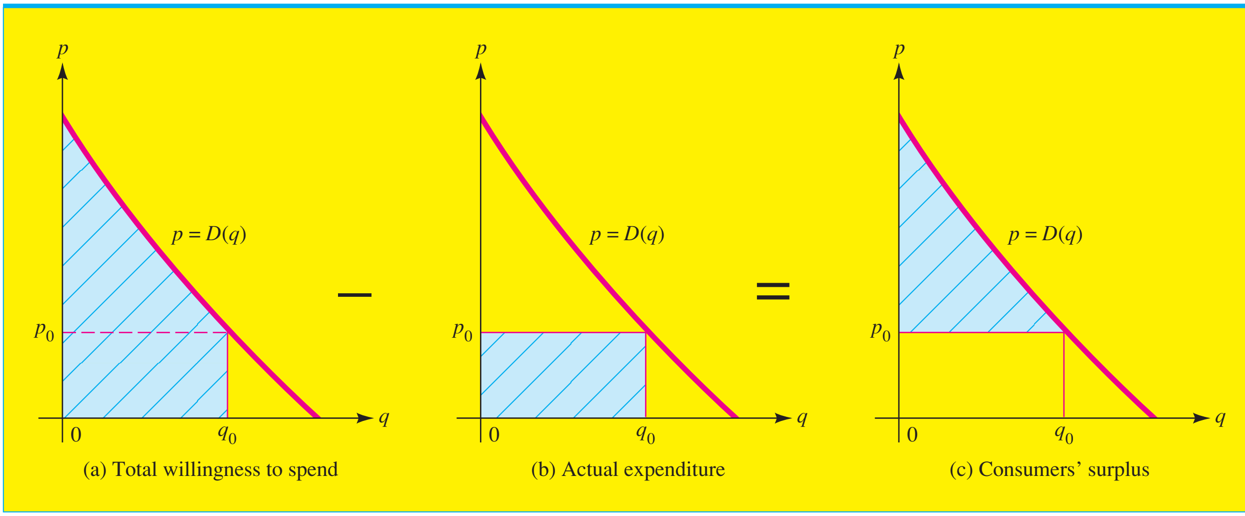
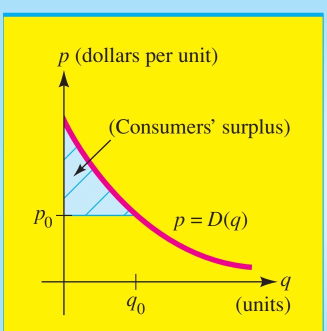



**FIGURE 6.15** Geometric interpretation of consumers' surplus.

Consumers' surplus has a simple geometric interpretation, which is illustrated in Figure 6.15. The symbols $p_0$ and $q_0$ denote the market price and corresponding demand, respectively. Figure 6.15a shows the region under the demand curve from $q = 0$ to $q = q_0$. Its area, as we have seen, represents the total amount that consumers are willing to spend to get $q_0$ units of the commodity. The rectangle in Figure 6.15b has an area of $p_0q_0$ and hence represents the actual consumer expenditure for $q_0$ units at $p_0$ dollars per unit. The difference between these two areas (Figure 6.15c) represents the consumers' surplus. That is, consumers' surplus CS is the area of the region between the demand curve $p = D(q)$ and the horizontal line $p = p_0$ and hence is equal to the definite integral



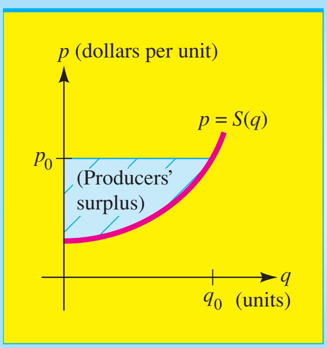

**Consumers' Surplus** ■ If $q_0$ units of a commodity are sold at a price of $p_0$ per unit and if $p = D(q)$ is the consumers' demand function for the commodity, then

$$\begin{bmatrix}\text{Consumers'} \\ \text{surplus}\end{bmatrix} = \begin{bmatrix}\text{total amount consumers} \\ \text{are willing to spend} \\ \text{for } q_0 \text{ units}\end{bmatrix} - \begin{bmatrix}\text{actual consumer} \\ \text{expenditure} \\ \text{for } q_0 \text{ units}\end{bmatrix}$$

$$\text{CS} = \int_0^{q_0} D(q)\,dq - p_0q_0$$

$$\begin{aligned}\text{CS} &= \int_0^{q_0} [D(q) - p_0]\, dq = \int_0^{q_0} D(q)\, dq - \int_0^{q_0} p_0\, dq \\ &= \int_0^{q_0} D(q)\, dq - p_0 q \Big|_0^{q_0} \\ &= \int_0^{q_0} D(q)\, dq - p_0 q_0\end{aligned}$$

**Producers' surplus** is the other side of the coin from consumers' surplus. In particular, a **supply function** $p = S(q)$ gives the price per unit that producers are willing to accept in order to supply $q_0$ units to the marketplace. However, any producer who is willing to accept less than $p_0 = S(q_0)$ dollars for $q_0$ units gains from the fact that the price is $p_0$. Then producers' surplus is the difference between what producers would be willing to accept for supplying $q_0$ units and the price they actually receive. Reasoning as we did with consumers' surplus, we obtain the following formula for producers' surplus.




**Producers' Surplus** ■ If $q_0$ units of a commodity are sold at a price of $p_0$ dollars per unit and $p = S(q)$ is the producers' supply function for the commodity, then

$$\begin{bmatrix}\text{Producers'} \\ \text{surplus}\end{bmatrix} = \begin{bmatrix}\text{actual consumer} \\ \text{expenditure} \\ \text{for } q_0 \text{ units}\end{bmatrix} - \begin{bmatrix}\text{total amount producers} \\ \text{receive when } q_0 \\ \text{units are supplied}\end{bmatrix}$$

$$\text{PS} = p_0 q_0 - \int_0^{q_0} S(q)\, dq$$

## EXAMPLE 2.7

A tire manufacturer estimates that $q$ (thousand) radial tires will be purchased (demanded) by wholesalers when the price is

$$p = D(q) = -0.1q^2 + 90$$

dollars per tire, and the same number of tires will be supplied when the price is

$$p = S(q) = 0.2q^2 + q + 50$$

dollars per tire.

**(a)** Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price.
**(b)** Determine the consumers' and producers' surplus at the equilibrium price.

**Solution**

**(a)** The supply and demand curves are shown in Figure 6.16. Supply equals demand when

$$\begin{aligned} -0.1q^2 + 90 &= 0.2q^2 + q + 50 \\ 0.3q^2 + q - 40 &= 0 \\ q &= 10 \quad (\text{reject } q = -13.33) \end{aligned}$$

and $p = -0.1(10)^2 + 90 = 80$ dollars per tire. Thus, equilibrium occurs at a price of \$80 per tire, where 10,000 tires are supplied and demanded.

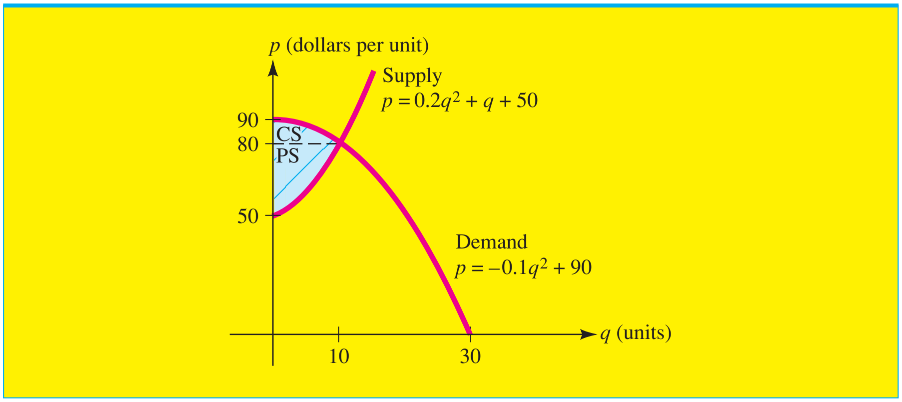



**FIGURE 6.16** Consumers' surplus and producers' surplus for the demand and supply functions in Example 2.7.

**(b)** Using $p_0 = 80$ and $q_0 = 10$, we find that the consumers' surplus is

$$\begin{aligned} \text{CS} &= \int_0^{10} (-0.1q^2 + 90)\, dq - (80)(10) \\ &= \left[ -0.1\left(\frac{q^3}{3}\right) + 90q \right]\Bigg|_0^{10} - (80)(10) \\ &= 866.67 - 800 = 66.67 \end{aligned}$$

or \$66,670 (since $q_0 = 10$ is really 10,000). The consumers' surplus is the area of the shaded region labeled CS in Figure 6.16.

The producers' surplus is

$$\begin{aligned}\text{PS} &= (80)(10) - \int_0^{10} (0.2q^2 + q + 50)\,dq \\ &= (80)(10) - \left[0.2\left(\frac{q^3}{3}\right) + \left(\frac{q^2}{2}\right) + 50q\right]\Bigg|_0^{10} \\ &= 800 - 556.67 = 243.33\end{aligned}$$

or \$243,330. The producers' surplus is the area of the shaded region labeled PS in Figure 6.16.

## P . R . O . B . L . E . M . S 6.2

**DEPRECIATION**

**1.** The resale value of a certain industrial machine decreases over a 10-year period at a rate that changes with time. When the machine is $x$ years old, the rate at which its value is changing is $220(x - 10)$ dollars per year. By how much does the machine depreciate during the second year?

**ADMISSIONS TO EVENTS**

**2.** The promoters of a county fair estimate that $t$ hours after the gates open at 9:00 A.M. visitors will be entering the fair at the rate of $-4(t + 2)^3 + 54(t + 2)^2$ people per hour. How many people will enter the fair between 10:00 A.M. and noon?

**MARGINAL COST**

**3.** At a certain factory, the marginal cost is $6(q - 5)^2$ dollars per unit when the level of production is $q$ units. By how much will the total manufacturing cost increase if the level of production is raised from 10 to 13 units?

**OIL PRODUCTION**

**4.** A certain oil well that yields 400 barrels of crude oil a month will run dry in 2 years. The price of crude oil is currently \$18 per barrel and is expected to rise at a constant rate of 3 cents per barrel per month. If the oil is sold as soon as it is extracted from the ground, what will be the total future revenue from the well?

**FARMING**

**5.** It is estimated that $t$ days from now a farmer's crop will be increasing at the rate of $0.3t^2 + 0.6t + 1$ bushels per day. By how much will the value of the crop increase during the next 5 days if the market price remains fixed at \$3 per bushel?

**SALES REVENUE**

**6.** It is estimated that the demand for a manufacturer's product is increasing exponentially at the rate of 2% per year. If the current demand is 5,000 units per year and if the price remains fixed at \$400 per unit, how much revenue will the manufacturer receive from the sale of the product over the next 2 years?

**EFFICIENCY**

**7.** After $t$ hours on the job, a factory worker can produce $100te^{-0.5t}$ units per hour. How many units does a worker who arrives on the job at 8:00 A.M. produce between 10:00 A.M. and noon?

**INVESTMENT**

**8.** Suppose that $t$ years from now, one investment plan will be generating profit at the rate of $P_1'(t) = 100 + t^2$ hundred dollars per year, while a second investment will be generating profit at the rate of $P_2'(t) = 220 + 2t$ hundred dollars per year.

**(a)** For how many years does the rate of profitability of the second investment exceed that of the first?

**(b)** Compute the net excess profit assuming that you invest in the second plan for the time period determined in part (a).

**(c)** Sketch the rate of profitability curves $y = P_1'(t)$ and $y = P_2'(t)$ and shade the region whose area represents the net excess profit computed in part (b).

**INVESTMENT** **9.** Answer the questions in Problem 8 for two investments with respective rates of profitability $P_1'(t) = 130 + t^2$ hundred dollars per year and $P_2'(t) = 306 + 5t$ hundred dollars per year.

**INVESTMENT** **10.** Answer the questions in Problem 8 for two investments with respective rates for profitability $P_1'(t) = 60e^{0.12t}$ thousand dollars per year and $P_2'(t) = 160e^{0.08t}$ thousand dollars per year.

**INVESTMENT** **11.** Answer the questions in Problem 8 for two investments with respective rates of profitability $P_1'(t) = 90e^{0.1t}$ thousand dollars per year and $P_2'(t) = 140e^{0.07t}$ thousand dollars per year.

**NET EARNINGS** **12.** Suppose that when it is $t$ years old, a particular industrial machine generates revenue at the rate $R'(t) = 6{,}025 - 8t^2$ dollars per year and that operating and servicing costs accumulate at the rate $C'(t) = 4{,}681 + 13t^2$ dollars per year.

**(a)** How many years pass before the profitability of the machine begins to decline?

**(b)** Compute the net earnings generated by the machine over the time period determined in part (a).

**(c)** Sketch the revenue rate curve $y = R'(t)$ and the cost rate curve $y = C'(t)$ and shade the region whose area represents the net earnings computed in part (b).

**NET EARNINGS** **13.** Answer the questions in Problem 12 for a machine that generates revenue at the rate $R'(t) = 7{,}250 - 18t^2$ dollars per year and for which costs accumulate at the rate $C'(t) = 3{,}620 + 12t^2$ dollars per year.

**EFFICIENCY** **14.** After $t$ hours on the job, one factory worker is producing $Q_1(t) = 60 - 2(t - 1)^2$ units per hour, while a second worker is producing $Q_2(t) = 50 - 5t$ units per hour.

**(a)** If both arrive on the job at 8:00 A.M., how many more units will the first worker have produced by noon than the second worker?

**(b)** Interpret the answer in part (a) as the area between two curves.

**FUND-RAISING** **15.** It is estimated that $t$ weeks from now, contributions in response to a fund-raising campaign will be coming in at the rate of $R'(t) = 5{,}000e^{-0.2t}$ dollars per week, while campaign expenses are expected to accumulate at the constant rate of \$676 per week.

**(a)** For how many weeks does the rate of revenue exceed the rate of cost?

**(b)** What net earnings will be generated by the campaign during the period of time determined in part (a)?

**(c)** Interpret the net earnings in part (b) as an area between two curves.

**FUND-RAISING** **16.** Answer the questions in Problem 15 for a charity campaign in which contributions are made at the rate of $R'(t) = 6{,}537e^{-0.3t}$ dollars per week and expenses accumulate at the constant rate of \$593 per week.

**THE AMOUNT OF AN INCOME STREAM** **17.** Money is transferred continuously into an account at the constant rate of \$2,400 per year. The account earns interest at the annual rate of 6% compounded continuously. How much will be in the account at the end of 5 years?

**THE AMOUNT OF AN INCOME STREAM** **18.** Money is transferred continuously into an account at the constant rate of \$1,000 per year. The account earns interest at the annual rate of 10% compounded continuously. How much will be in the account at the end of 10 years?

**THE PRESENT VALUE OF AN INVESTMENT** **19.** An investment will generate income continuously at the constant rate of \$1,200 per year for 5 years. If the prevailing annual interest rate remains fixed at 12% compounded continuously, what is the present value of the investment?

**THE PRESENT VALUE OF A FRANCHISE** **20.** The management of a national chain of pizza parlors is selling a 6-year franchise to operate its newest outlet in Orlando. Experience in similar localities suggests that $t$ years from now the franchise will be generating profit at the rate of $f(t) = 10{,}000 + 500t$ dollars per year. The prevailing rate of interest remains fixed during the next 6 years at 6% compounded continuously. What is the present value of the franchise?

**THE PRESENT VALUE OF A FRANCHISE** **21.** The management of a national chain of fast-food outlets is selling a 10-year franchise in Cleveland, Ohio. Past experience in similar localities suggests that $t$ years from now the franchise will be generating profit at the rate of $f(t) = 10{,}000 + 500t$ dollars per year. If the prevailing annual interest rate remains fixed at 10% compounded continuously, what is the present value of the franchise?

**CONSUMER'S WILLINGNESS TO SPEND** *For the consumers' demand functions* D(q) *in Problems 22 through 27:*

***(a)*** *Find the total amount of money consumers are willing to spend to get* $q_0$ *units of the commodity.*

***(b)*** *Sketch the demand curve and interpret the consumers' willingness to spend in part* (a) *as an area.*

**22.** $D(q) = 2(64 - q^2)$ dollars per unit; $q_0 = 6$ units

**23.** $D(q) = \dfrac{300}{(0.1q + 1)^2}$ dollars per unit; $q_0 = 5$ units

**24.** $D(q) = \dfrac{400}{0.5q + 2}$ dollars per unit; $q_0 = 12$ units

**25.** $D(q) = \dfrac{300}{4q + 3}$ dollars per unit; $q_0 = 10$ units

**26.** $D(q) = 40e^{-0.05q}$ dollars per unit; $q_0 = 10$ units

**27.** $D(q) = 50e^{-0.04q}$ dollars per unit; $q_0 = 15$ units

**CONSUMERS' SURPLUS** *In Problems 28 through 31,* p = D(q) *is the price (dollars per unit) at which* q *units of a particular commodity will be demanded by the market (that is, all* q *units will be sold at this price), and* $q_0$ *is a specified level of production. In each case, find the price*

$p_0 = D(q_0)$ *at which* $q_0$ *units will be demanded and compute the corresponding consumers' surplus CS. Sketch the demand curve* $y = D(q)$ *and shade the region whose area represents the consumers' surplus.*

**28.** $D(q) = 2(64 - q^2)$; $q_0 = 3$ units

**29.** $D(q) = 150 - 2q - 3q^2$; $q_0 = 6$ units

**30.** $D(q) = 40e^{-0.05q}$; $q_0 = 5$ units

**31.** $D(q) = 75e^{-0.04q}$; $q_0 = 3$ units

**PRODUCERS' SURPLUS** *In Problems 32 through 35,* $p = S(q)$ *is the price (dollars per unit) at which* $q$ *units of a particular commodity will be supplied to the market by producers, and* $q_0$ *is a specified level of production. In each case, find the price* $p_0 = S(q_0)$ *at which* $q_0$ *units will be supplied and compute the corresponding producers' surplus PS. Sketch the supply curve* $y = S(q)$ *and shade the region whose area represents the producers' surplus.*

**32.** $S(q) = 0.3q^2 + 30$; $q_0 = 4$ units

**33.** $S(q) = 0.5q + 15$; $q_0 = 5$ units

**34.** $S(q) = 10 + 15e^{0.03q}$; $q_0 = 3$ units

**35.** $S(q) = 17 + 11e^{0.01q}$; $q_0 = 7$ units

**CONSUMERS' SURPLUS** **36.** A manufacturer has determined that when $q$ units of a particular commodity are produced, the price at which all the units can be sold is $p = D(q)$ dollars per unit, where $D$ is the demand function given by

$$D(q) = \frac{300}{(0.1q + 1)^2}$$

**(a)** How many units can the manufacturer expect to sell if the commodity is priced at $p_0 = \$12$ per unit?

**(b)** Find the consumers' surplus that corresponds to the level of production $q_0$ found in part (a).

**CONSUMERS' SURPLUS** **37.** Answer the questions in Problem 36 for the demand function

$$D(q) = \frac{400}{0.5q + 2}$$

and price $p_0 = \$20$ per unit.

**CONSUMERS' SURPLUS** **38.** Parts for a piece of heavy machinery are sold by the manufacturer in units of 1,000, and $q$ such units will be sold when the price is $p = 110 - q$ dollars per unit. The total cost of producing those $q$ units is $C(q) = q^3 - 25q^2 + 2q + 3{,}000$ dollars.

**(a)** How much profit is derived from the sale of the $q$ units at $p$ dollars per unit? [*Hint:* profit = revenue − cost; how much revenue is derived from the sale of the $q$ units?]

**(b)** For what value of $q$ is profit maximized?

**(c)** Find the consumers' surplus for the level of production $q_0$ that corresponds to maximum profit.

**CONSUMERS' SURPLUS**

**39.** Repeat Problem 38 for $p = 124 - 2q$ and $C(q) = 2q^3 - 59q^2 + 4q + 7{,}600$.

**CONSUMERS' AND PRODUCERS' SURPLUS**

**40.** Suppose that $q$ units of a certain commodity are demanded by the market (that is, sold) when the price is $p = D(q)$ dollars per unit and that the same number of units are supplied by manufacturers when the price is $p = S(q)$ dollars per unit, where the demand and supply functions are, respectively,

$$D(q) = 110 - q^2 \qquad \text{and} \qquad S(q) = \frac{1}{3}q^2 + 2q + 5$$

**(a)** At what level of production $q_0$ does supply equal demand? This is called the **equilibrium level,** and the corresponding price $p_0$ is the **equilibrium price.**

**(b)** Compute the consumers' surplus and the producers' surplus at market equilibrium.

**(c)** First sketch the demand curve $y = 32 - q^2$ and the supply curve $y = \frac{1}{3}q^2 + 2q + 5$ on the same coordinate axes, and then shade and label the regions whose areas respectively correspond to consumers' surplus and producers' surplus at market equilibrium.

**CONSUMERS' AND PRODUCERS' SURPLUS**

**41.** Repeat Problem 40 for a commodity whose demand and supply functions are, respectively,

$$D(q) = \frac{16}{q + 2} - 3 \qquad \text{and} \qquad S(q) = \frac{1}{3}(q + 1)$$

**PRESENT VALUE**

**42.** A certain investment generates income continuously over a period of $N$ years. After $t$ years, the investment will be generating income at the rate of $f(t)$ dollars per year. Derive an expression for the present value of this investment if the prevailing annual interest remains fixed at $r$ (expressed as a decimal), compounded continuously.

**TOTAL REVENUE**

**43.** Consider the following problem: "A certain oil well that yields 300 barrels of crude oil a month will run dry in 3 years. It is estimated that $t$ months from now the price of crude oil will be $P(t) = 18 + 0.3\sqrt{t}$ dollars per barrel. If the oil is sold as soon as it is extracted from the ground, what will be the total future revenue from the well?"

**(a)** Solve this problem using antidifferentiation, as in Chapter 5.

**(b)** Solve the problem using definite integration. [*Hint:* Divide the 3-year (36 month) time interval $0 \le t \le 36$ into $n$ equal subintervals of length $\Delta t$ and let $t_j$ denote the beginning of the $j$th subinterval. Find an expression that estimates the revenue $R(t_j)$ obtained during the $j$th subinterval. Then express the total revenue as the limit of a sum.]

**(c)** Read an article on the petroleum industry and write a paragraph on mathematical methods of modeling oil production.*

* A good place to start is the article by J. A. Weyland and D. W. Ballew, "A Relevant Calculus Problem: Estimation of U.S. Oil Reserves," *The Mathematics Teacher,* Vol. 69 (1976), pp. 125–126.

**INVENTORY STORAGE COSTS**

**44.** A manufacturer receives $N$ units of a certain raw material that are initially placed in storage and then withdrawn and used at a constant rate until the supply is exhausted 1 year later. Suppose storage costs remain fixed at $p$ dollars per unit per year. Use definite integration to find an expression for the total storage cost the manufacturer will pay during the year.

[*Hint:* Let $Q(t)$ denote the number of units in storage after $t$ years and find an expression for $Q(t)$. Then subdivide the interval $0 \leq t \leq 1$ into $n$ equal subintervals and express the total storage cost as the limit of a sum.]

*In Problems 45 and 46 use the numeric integration feature of your calculator to compute the required quantity.*

**45.** An investment will generate income continuously at the constant rate of \$1,750 per year for 10 years. If the prevailing rate of interest remains fixed at 9.5% compounded continuously, what is the present value of the investment?

**46.** An investor is planning to buy a business that $t$ years from now is expected to be generating income at the rate of $f(t) = 5{,}000 + 300t + 1.7t^2$ dollars per year. If this pattern continues for the next 7 years and the prevailing rate of interest remains fixed at 8% compounded continuously, what is the present value of the investment?

# 3 Additional Applications of Definite Integration

In Section 6.2, we examined applications of definite integration to business and economics, and in this section, we extend our list of applications to areas such as biology, the social sciences, and medicine. We begin by showing how integration can be used to compute the average value of a function over an interval.

## Average Value of a Function

A teacher who wants to compute the average score on an examination simply adds all the individual scores and divides by the number of students taking the exam, but how should one go about finding, say, the average level of pollution in a city during the daytime hours? The difficulty is that since time is continuous, there are "too many" pollution levels to add up in the usual way, so how should we proceed?

We begin by subdividing the interval $a \leq x \leq b$ into $n$ parts, each of length $\Delta_n x = \dfrac{b - a}{n}$. If $x_j$ is a number taken from the $j$th subinterval for $j = 1, 2, \ldots, n$, then the average of the corresponding functional values $f(x_1), f(x_2), \ldots, f(x_n)$ is